6^{th} Singapore Physics League Grand Finals (Junior)

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Website: sgphysicsleague.org
Email: contact@sgphysicsleague.org

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Joey Tang Lead Developer	**Aaron Ng** Problem Setter	**E-jie Tan** Problem Setter	**Gabriel Poon** Problem Setter
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Alvin Liu Logistics	**Shanay Jindal** Logistics	**Jed Lim** Data Analyst	**Xu Tiange** Data Analyst

Luc Mazereeuw
Data Analyst, Problem Setter

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Problem List

Physical Mathematics (100 points)

Section A. Gas the area (25 points)

The diagram below is a trirectangular tetrahedron (right-angled tetrahedron) where all three face angles at one vertex (the apex) are right angles. It is given that 3 of the faces have known area A, B, and C respectively as labelled below.

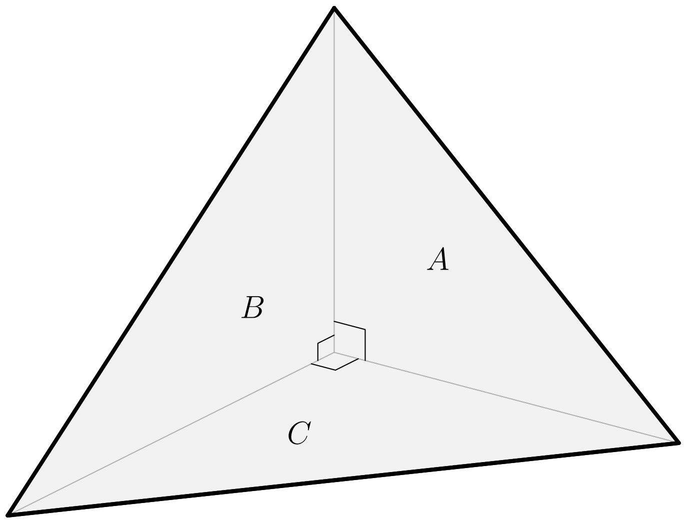


Figure 1: A trirectangular tetrahedron.

A.1	Now, we fill our tetrahedron with compressed gas! Let us denote the pressure force acting on each face vectorially as $\vec{F}_A$, $\vec{F}_B$, $\vec{F}_C$ and $\vec{F}_X$, where X is used to denote the last face with unknown area. Express $\vec{F}_X$ in terms of the other 3 forces. *Leave your answer in the simplest possible form in terms of $\vec{F}_A$, $\vec{F}_B$, and $\vec{F}_C$.*	5 pt

It is given that the **dot product** between 2 vectors $\vec{a}$ and $\vec{b}$ (also written as $\vec{a} \cdot \vec{b}$) is 0 if the vectors are perpendicular.

A.2	Using this idea of the dot product, find a vectorial relationship between the forces acting on the 3 known faces $\vec{F}_A$, $\vec{F}_B$ and $\vec{F}_C$. Make use of all three vectors in your relationship. There are multiple correct answers. *Leave your answer as an expression in terms of $\vec{F}_A$, $\vec{F}_B$, and $\vec{F}_C$.*	5 pt

We know that for a right-angled triangle with the lengths of opposite and adjacent sides being a and b, the length of the hypotenuse is $c^2 = a^2 + b^2$.

A.3	Given that the pressure is uniform in the tetrahedron and making use of your answers in **A.1** and **A.2**, find the area of the unknown face X. *Leave your answer in the simplest possible form in terms of* A, B, *and* C.	15 pt

Hint: You are **not** advised to bash out the answer mathematically. Try to think of a physical way to do it.

Section B. Hooke'd on inequalities **(30 points)**

In this question, we will consider combinations of series and parallel springs. Consider n^2 springs with spring constants $k_1, k_2, ..., k_n$, with $k_1 > k_2 > k_3 > ... > k_n$. Assume that all springs have 0 rest length. These springs are arranged in two different configurations (not drawn to scale).

- **Configuration 1:** The springs are arranged in a grid without additional connections between them
- **Configuration 2:** The springs are arranged in the same grid, but rigid vertical wooden boards are attached, connecting the springs within each column so that they move together as a unit.

In both cases, let the length of the spring with spring constant k_i in the **leftmost** column be x_i, where $i \in (1, 2, ..., n)$.

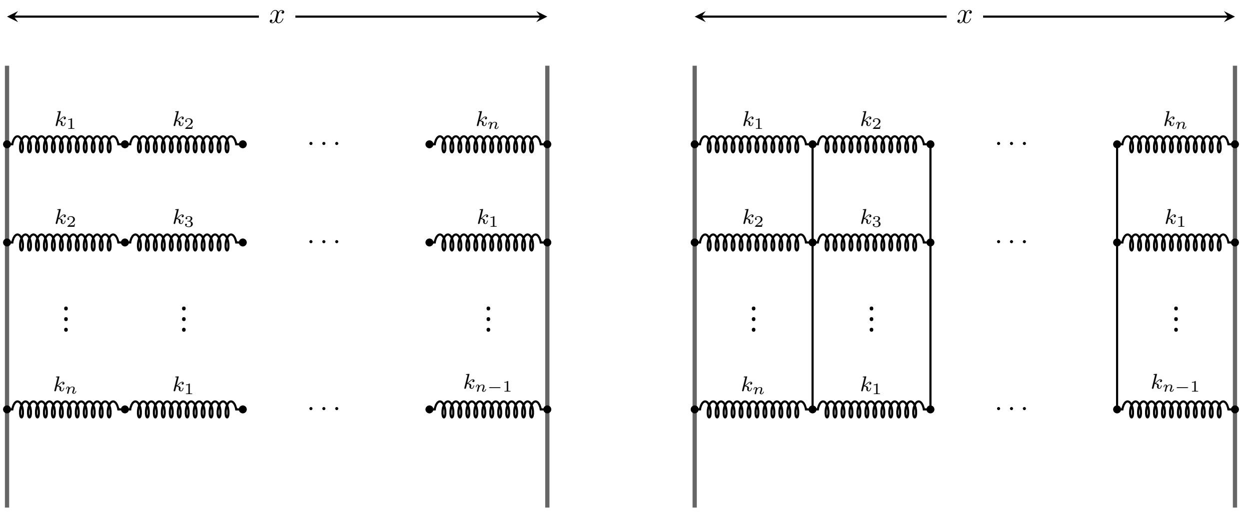


Figure 2: (Left) Configuration 1 of springs. (Right) Configuration 2 of springs. All springs are initially set to have the same length and we let the system settle into equilibrium.

For both configurations, the left end is fixed in place while the right end is moved a length x away.

B.1	Find (i) the value of $x_1 + x_2 + \cdots + x_n$ in **configuration 1** in terms of x, (ii) the value of x_1 in **configuration 2** in terms of x, and (iii) a non-trivial function $f(x_i, k_i)$ that is constant for all values of i in **configuration 1**. *Leave your answers as 3 expressions in the simplest possible form in terms of the given variables.*	10 pt

B.2	Hence or otherwise, deduce which configuration requires a larger force for the same extension. *Your answer should be either "Configuration 1" or "Configuration 2". You are only given a single attempt for this subpart.*	5 pt

The force required to stretch each configuration by a length x can be written as $F = k_{\text{eff}}x$, where k_{eff} is the effective spring constant of the system.

B.3	Find expressions of k_{eff} for configurations 1 and 2. You may use the summation notation $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$. *Leave your answers as 2 expressions in the simplest possible form in terms of $k_1, k_2, \cdots, k_n$ and n.*	10 pt

B.4	Hence, using your answers in B.2 and B.3, find a non-trivial inequality in either the form $f(k_1, k_2, \cdots, k_n) < n^2$ or $f(k_1, k_2, \cdots, k_n) > n^2$, where f is a function of the spring constants only. You may use the summation notation $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$. *Leave your answer in terms of $k_1, k_2, \cdots, k_n$ and n.*	5 pt

Do you recognise this inequality?

Section C. This problem has potential (45 points)

Given three points A, B, and C in space, consider the sum of distances $\overline{AX} + \overline{BX} + \overline{CX}$ to a point X. What point X minimizes this sum?

To the more mathematically inclined individuals among you, you may recognise the point X to be the **Fermat Point** of the triangle formed by ABC. But we are not doing pure mathematics here.

The Hungarian mathematician George Pólya modelled this famous geometric problem using a physical system consisting of a horizontal plane with 3 holes. 3 ropes are passed through the holes, each attached to an equal mass m at one end and a common mass at the other. If all three ropes have the same length L, and we define the height of the horizontal plane to have zero gravitational potential, then the potential energy of the system ϕ is

$$\phi = mg(\overline{AX} + \overline{BX} + \overline{CX} - 3L)$$

At mechanical equilibrium, the potential energy of the system is minimised, and hence $\overline{AX} + \overline{BX} + \overline{CX}$ is minimised. Since the tensions are equal, the angles between the force vectors at X are equal to 120 degrees, and this condition allows us to locate the Fermat Point (if it exists within the triangle).

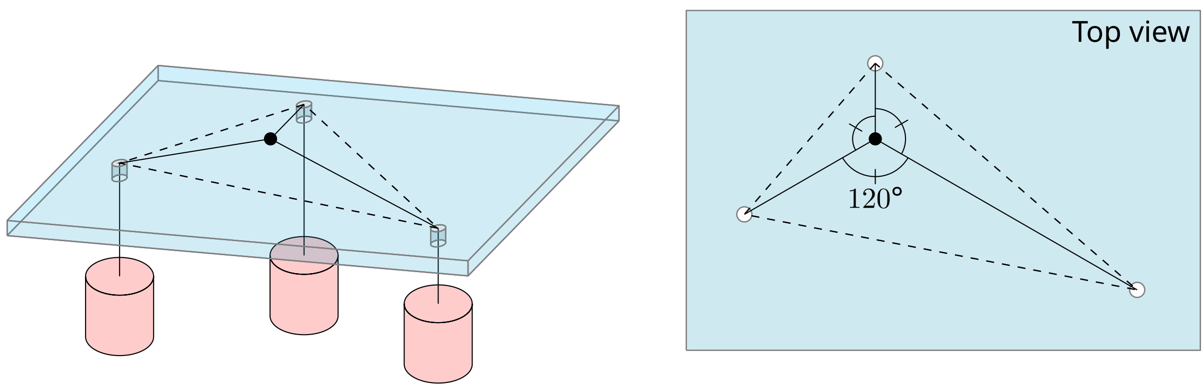
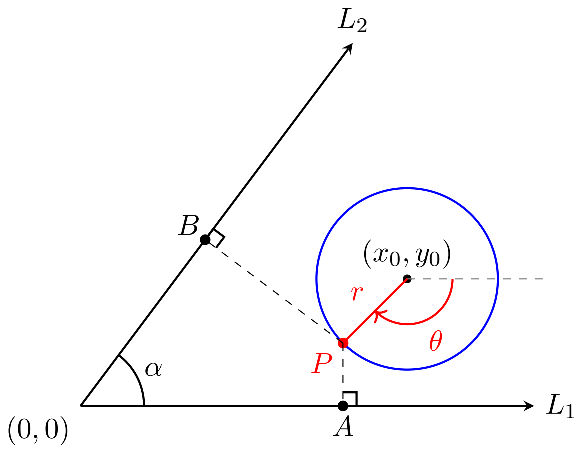


Figure 3: A mechanical proof of the Fermat point.

Let us now tackle some similar problems using the same strategy.

C.1	Given 3 points $A = (-11, 10)$, $B = (67, 23)$ and $C = (-21, 44)$, find the co-ordinates (x, y) of the point X such that $\overline{AX}^2 + \overline{BX}^2 + \overline{CX}^2$ is minimised. *Leave your answer to 3 significant figures.*	10 pt

Hint: Imagine connecting X to each of the 3 points with Hookean springs.



C.2	A circle lies inside an acute angle. Let point P be the point on the circle where the sum of distances $\overline{PA}+\overline{PB}$ to the angle sides is minimal. Find the value of θ for this point P. *Leave your answer in the simplest possible form in terms of any or all of α, x_0, y_0 and r.*	15 pt

Hint: Consider a similar set-up to the mechanical Fermat point proof shown above.

Finally, we will show a surprising use-case for this technique. In mathematics, the moving sofa problem or sofa problem is a two-dimensional idealisation of real-life furniture-moving problems. It asks for the largest area of a rigid 2D shape that can be manoeuvred through an L-shaped planar region (a hallway) with legs of unit width.

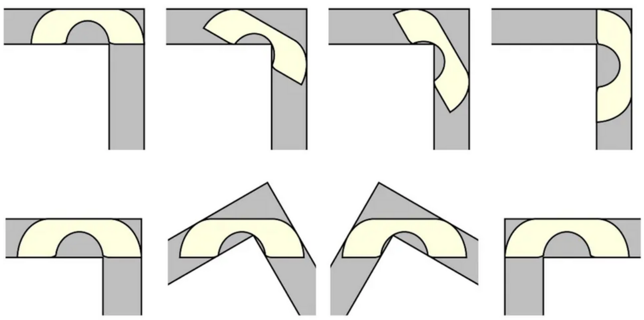
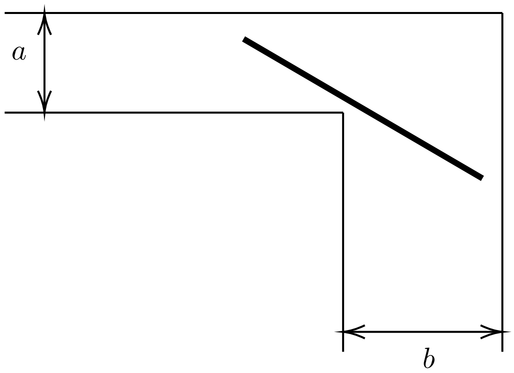


Figure 4: Moving a sofa

The area thus obtained is referred to as the sofa constant. However, the exact value of the sofa constant for an arbitrary hallway is an open problem. In this problem, we will consider a simpler variation where we try to manoeuvre a straight ladder across a corner, and solve it with physics!



C.3 Two hallways of widths a and b meet at a right angle. What is the length l of the longest ladder that can be passed around the corner from one hallway to the other? 20 pt

Leave your answer in the simplest possible form in terms of a *and* b*.*

Hint: Imagine the ladder is a straight, unbendable spring under extension

Cable Management (100 points)

Section A. Coaxial Cables (35 points)

In this problem, we study physical scenarios involving three types of cables commonly used in networking: coaxial cables, twisted pairs and fibre-optic cables.

A coaxial cable consists of an inner conducting core surrounded by a concentric conducting shield, separated by a dielectric. As opposed to an unshielded wire, an ideal coaxial cable can be installed next to metal objects without power losses.

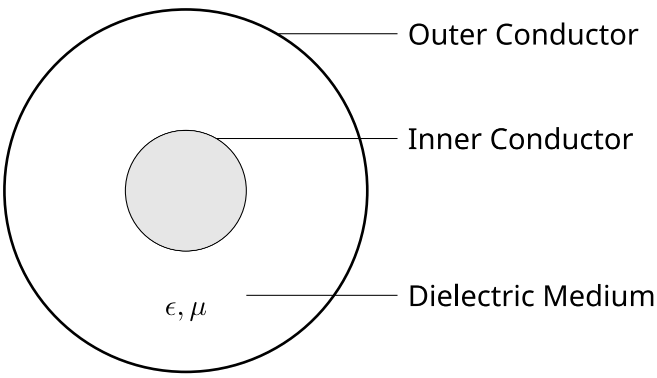
Suppose we have a long wire situated along the x-axis, carrying current I in the $+x$-direction. A small circular wire loop of resistance R and radius $s \ll D$ parallel to the xy-plane has its centre on the y-axis at a distance D from the origin.

A.1	Determine the magnitude of the magnetic flux Φ through the loop. You may use all relevant approximations. *Leave your answer in the simplest possible form in terms of* I, D, s *and* μ_0.	5 pt

The current is now increased from zero at a constant rate α, i.e. the current I as a function of time t is $I(t) = \alpha t$.

A.2	Determine the power P dissipated in the loop during this process. *Leave your answer in the simplest possible form in terms of* α, D, s, R *and* μ_0.	10 pt

Now, suppose that the wire is instead replaced by a coaxial cable. The coaxial cable has a solid cylindrical core of radius a surrounded by a thin cylindrical shell of radius b, separated by a dielectric of permittivity ϵ and permeability μ. The core is charged with linear charge density λ and carries a current I travelling in the $+x$-direction. Both charge and current are uniformly distributed on its outer surface. By design, the shield also carries linear charge density $-\lambda$ and current I travelling in the $-x$-direction, both uniformly distributed on its *inner* surface.[1]



[1] Signals in coaxial cables are actually transmitted in the form of propagating *waves*, as opposed to a uniform charge and current across the whole cable.

A.3 Determine the magnitude of the 10 pt
(i) electric field $E(r)$ and
(ii) magnetic field $B(r)$
in the coaxial cable as a function of the radial distance r from its axis, in the region $a < r < b$.

Leave your answers as 2 expressions in the simplest possible form in terms of r, λ, I, ϵ and μ.

It can also be determined that in the region outside the coaxial cable, even though it carries both charge and current, the electric and magnetic fields are zero everywhere! This is how coaxial cables can minimise power losses even when installed next to conductors.

Since there exists both an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ between the core and the shield, energy is transported down the wire. The rate of energy flow can be determined using the **Poynting vector**, denoted by $\mathbf{S}$, which describes the power *per unit area* transported by the fields as

$$\mathbf{S} = \frac{\mathbf{E} \times \mathbf{B}}{\mu}$$

A.4 Suppose that $b = a + \Delta r$ where $\Delta r \ll a$. Determine the power P_T transported down the cable to lowest order in Δr. 10 pt

Leave your answer in the simplest possible form in terms of λ, I, a, Δr, and ϵ.

Hint 1: In the formula for the Poynting vector, $\times$ refers to the **cross product**, defined as

$$|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}| \sin\theta$$

where $\mathbf{A}$ and $\mathbf{B}$ are vectors, and the angle between them is θ.
Hint 2: Since $\Delta r \ll a$, the electric and magnetic fields between the core and the shield are essentially uniform.

Notice that even though the net current flowing through the cable is zero ($+I$ in the core and $-I$ in the shield), there is still a net power transported down the cable! This is in contrast to a normal wire which uses a net current to carry a net power.

Section B. Twisted Pairs (30 points)

A twisted pair is another type of cable, where two wires in a single circuit are twisted together. Twisted pairs are usually cheaper and have lower densities have coaxial cables. Twisting a pair of wires helps to reduce the electromagnetic radiation emitted by the pair compared to if they were untwisted.

Suppose we have a circuit consisting of two long wires parallel to the x-axis. One passes through the point $(0, a, 0)$ and carries current I in the $+x$-direction, while the other passes through the point $(0, -a, 0)$ and carries current I in the $-x$-direction. Note that coordinates are given in (x, y, z) triplets.

B.1 When $y \gg a$, the magnitude of the magnetic field on the y-axis can be approximately expressed as — 5 pt

$$B(y) \approx \frac{C_y}{y^2}$$

where C_y is some constant. Determine C_y.

Leave your answer in the simplest possible form in terms of I, a and μ_0.

B.2 When $z \gg a$, the magnitude of the magnetic field on the z-axis can be approximately expressed as — 10 pt

$$B(z) \approx \frac{C_z}{z^2}$$

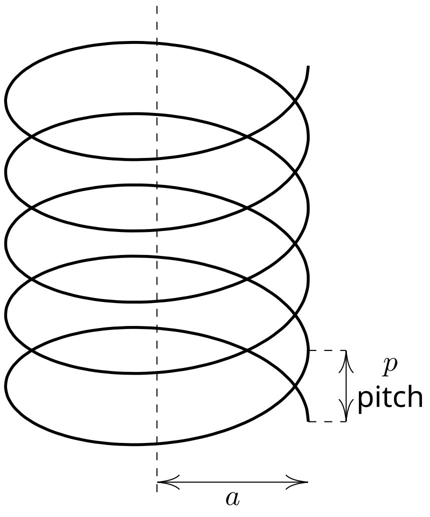
where C_z is some constant. Determine C_z.

Leave your answer in the simplest possible form in terms of I, a and μ_0.

Mathematical Hint: You may find the following approximation for $|x| \ll 1$ useful for the above:

$$\frac{1}{1-x} \approx 1 + x$$

Now, the two wires are twisted to form a twisted pair. As a result, the two wires each form solenoids carrying current I in opposite directions, centred on the x-axis and having opposite windings. Each solenoid takes the shape of a helix with radius a and pitch p, as shown below.



B.3	Determine the magnitude of the magnetic field B on the x-axis due to **each** solenoid. *Leave your answer in the simplest possible form in terms of I, p and μ_0.*	5 pt

B.4	When $r \gg a$, the magnitude of the magnetic field at a point a radial distance r from the x-axis due to **each** solenoid can be approximately expressed as $$B(r) \approx \frac{C_r}{r}$$ where C_r is some constant. Determine C_r. *Leave your answer in the simplest possible form in terms of I, a, p and μ_0.*	10 pt

Hint: At large distances, the magnetic field due to the *winding* of the solenoid can be neglected.

Since both solenoids have opposite windings and opposite currents, their magnetic fields outside the solenoids approximately cancel, leading to an *exponential decay* in the total magnetic field with distance far away from the twisted pair. This illustrates the effect of twisting wires together in reducing the field around the pair, as compared to an untwisted pair.

Section C. Fibre-Optic Cables (35 points)

Fibre-optic cables are yet another type of cable which use optical fibres (consisting of a core and a cladding layer) to carry light instead of electricity. Fibre-optic cables typically have the most data throughput and can cover the largest distances, while also being least prone to interference as they use photons instead of charges to carry signals.

Suppose we have a fibre optic cable with a cylindrical core of radius $r = 0.625$ mm and refractive index $n_1 = 1.61$, surrounded by a coaxial layer of cladding with refractive index $n_2 = 1.49$. A light ray enters the cable and remains entirely confined within the core as it travels along it.

C.1	Inside the cable, determine the maximum angle θ_{max} the light ray can make with the axis of the cable such that it does not escape the core. *Leave your answer to 1 decimal place in units of degrees.*	4 pt

The cable is first stretched to be completely straight. A signal enters the core from a point on the axis in the form of many light rays, where the angles they make with the axis of the cable range from a minimum of $0°$ (parallel to the axis) to a maximum of θ_{max} (the maximum allowable angle found in C.1). The signal is instantaneous (has zero duration) at the input and the length of the cable is $D = 2.40$ km.

C.2	Determine the duration t of the signal at the output. *Leave your answer to 3 significant figures in units of* μs.	8 pt

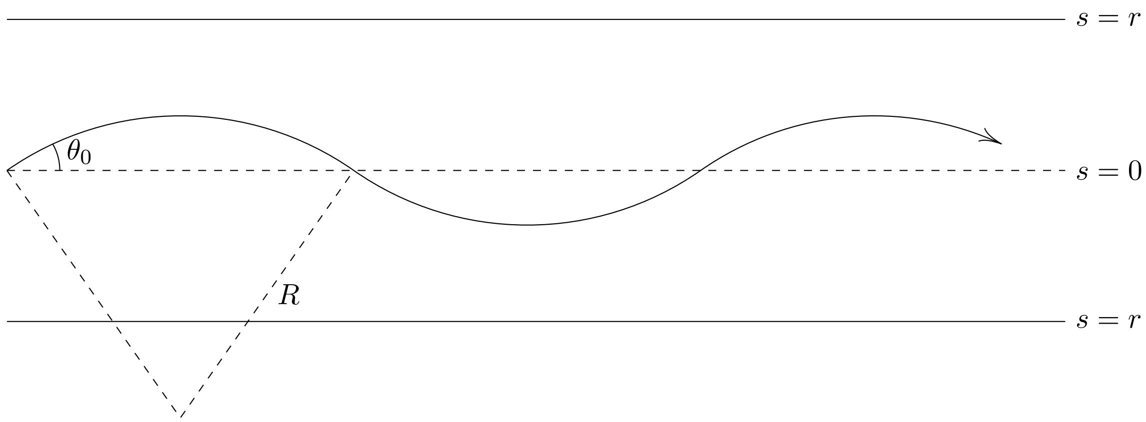
This highlights a phenomenon known as **modal dispersion**, and it causes signals within a fibre-optic cable to spread out and distort. Practically, this means that fibre-optic cables have a limited bandwidth unless resolved.

One common method to combat this issue is the use of a **graded-index fibre**. As opposed to a step-index fibre, which consists of sharp decreases in refractive index between interfaces, a graded-index fibre has a core whose refractive index decreases continuously with increasing radius. As a result, light rays curve in the fibre and are repeatedly brought back to the optical axis, decreasing modal dispersion.

Suppose a core has a refractive index n that depends on the distance s to the optical axis as

$$n(s) = \frac{n_1}{1 + \frac{s}{r}\left(\frac{n_1}{n_2} - 1\right)}$$

surrounded by air of refractive index $n_0 = 1$. A light ray enters the core of the cable through its centre such that after it has entered the cable, the ray makes an angle of θ_0 with the optical axis. It turns out that the trajectory of the light ray within the core is a series of circular arcs of constant radius R.



C.3	Determine R. *Leave your answer in terms of* r*,* θ_0*,* n_1 *and* n_2*.*	15 pt

Hint: Let n be the refractive index and θ be the angle the ray makes with the axis. Snell's law implies that $n \sin(90° - \theta) = n \cos\theta$ is constant along the trajectory of the light ray. Construct a trigonometric relation using the fact that the trajectory is a circular arc.

C.4	Inside the cable, determine the maximum initial angle $\theta_{0,\text{max}}$ the light ray can make with the axis of the cable such that it does not escape the core. *Leave your answer to 1 decimal place in units of degrees.*	8 pt

Chaos Ball Theory (100 points)

Section A. Up and Down (25 points)

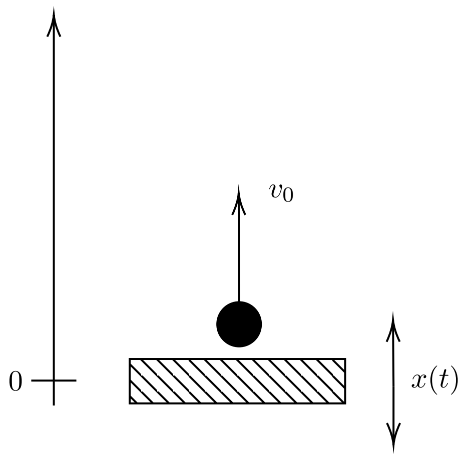
A horizontal piston sinusoidally oscillates in the vertical direction with frequency ω and amplitude x_0. We can express the vertical coordinate x of the piston in terms of time t as

$$x(t) = x_0 \cos \omega t$$

By differentiating the position of the piston with respect to time, we obtain that at time t, the piston is moving at velocity

$$v(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(x_0 \cos \omega t\right) = -\omega x_0 \sin \omega t$$

A ball is initially launched upwards with speed v_0 from the vertical coordinate $x = 0$ and free falls under Earth's gravitational field g directed downwards. (Let us neglect the fact that the ball starts under the piston — suppose it passes through a hole in the piston that is quickly sealed afterwards.) The ball then collides with and bounces off the piston elastically.



A.1	Determine an equation satisfied by t_0, the time t when the ball first bounces off the piston. You do not need to solve the equation explicitly for t_0. *Leave your answer as an equation in the simplest form in terms of t_0, v_0, x_0, ω and g.*	3 pt

We shall attempt to parametrise the system with a recurrence equation in t_n, the time of the $(n+1)^{\text{th}}$ bounce. In order to do so, we will make some simplifying assumptions.

A.2 Suppose we approximate the speed of the ball right before it bounces as a constant v_0.[a] Since the piston is moving and the collision is elastic, the speed of the ball changes after the bounce. Determine the new speed v of the ball immediately after the bounce that occurs at time t. — 8 pt

Leave your answer in terms of t, v_0, x_0, ω *and* g.

[a] This assumption is atypical, but we use it to obtain a one-dimensional chaotic map which will greatly simplify our following analyses.

A.3 Suppose we also approximate the time interval between consecutive bounces to be twice the time taken to reach the maximum height. Determine a recurrence equation for t_{n+1} in terms of t_n. — 8 pt

Leave your answer in the simplest form in terms of t_n, v_0, x_0, ω *and* g.

It turns out that this simple mechanical system can exhibit **chaos**. The most commonly known property of a chaotic dynamical system is that it is sensitive to initial conditions, which means that a very small change in the initial conditions can result in very large differences in the future, even though the system is deterministic.

The reason this system exhibits chaos is because its recurrence equations are actually equivalent to a chaotic map known as the **circle map**, defined by the recurrence equation

$$\theta_{n+1} = \theta_n + \Omega - \frac{K}{2\pi}\sin 2\pi\theta_n$$

under the variable θ and for some constants Ω and K. (θ is actually taken modulo 1, but we shall neglect that for now.)

A.4 Express θ_n in terms of t_n such that the system can be characterised by the circle map, and deduce the constants Ω and K. — 6 pt

Leave your answers as 3 expressions in the simplest form in terms of t_n, v_0, x_0, ω *and/or* g.

Section B. Double Trouble (35 points)

We shall now deviate from the physical model (which is bound by the crude approximations we made) and begin to study the circle map purely mathematically.

The circle map is defined by the recurrence equation $\theta_{n+1} = f(\theta_n)$ where

$$f(\theta) = \left(\theta + \Omega - \frac{K}{2\pi}\sin 2\pi\theta\right) \mod 1$$

i.e. $0 \leq \theta < 1$. Here, $\mathrm{mod}\ 1$ denotes that we take the decimal part of the result (or one minus the decimal part for negative values).

For certain values of the parameters K, Ω as well as the initial condition θ_0, the map is periodic, i.e. the value of θ begins to repeat after some number of iterations. We shall investigate the period of this map under different parameters and conditions.

First, let us suppose that $K = \frac{\pi}{2}$ and $\Omega = \frac{1}{8}$. This map has period 1 for almost all initial conditions θ_0, which means that θ eventually approaches some fixed value θ_∞ that we call the **fixed point**. This fixed point is *stable*, since an initial condition near the fixed point will eventually converge to it. It can be shown that a fixed point is stable if and only if the slope of the map at that point has magnitude at most one. The slope of the map at some θ is given by its derivative with respect to θ:

$$f'(\theta) = 1 - K\cos 2\pi\theta$$

B.1	For $K = \frac{\pi}{2}$ and $\Omega = \frac{1}{8}$, determine the value of θ_∞. *If the answer is exact, leave your answer in an exact form. Otherwise, leave your answer to 3 significant figures.*	6 pt

If we increase the value of K, there is a critical value K_{crit} where the fixed point becomes *unstable*, and an initial condition close to the fixed point will eventually diverge away from it. This results in a doubling of the period of the map to 2, i.e. θ eventually approaches an alternating sequence of two fixed values. This is a process called **bifurcation**.

B.2	For $\Omega = \frac{1}{8}$, determine the value of K_{crit}. *If the answer is exact, leave your answer in an exact form. Otherwise, leave your answer to 3 significant figures.*	12 pt

Hint: As we *increase* the value of K past its initial value, when is the *first* time that the slope has magnitude greater than one?

Mathematical Hint: It may be useful to know that for $-1 \leq x \leq 1$,

$$\cos(\sin^{-1} x) = \sqrt{1 - x^2}$$

It turns out that as we continue increase the value of K, this process will repeat at an exponentially increasing rate, with the period doubling to 4, 8, 16, ... and so on. This is sometimes called the **route to chaos**. Finally, at some limiting value of K, chaos begins and the map becomes aperiodic, with its values no longer approaching a periodic cycle.

We can study this behaviour by numerically solving the recurrence equation. This task is typically carried out by computers by iterating the recurrence equation over thousands or millions of times. Luckily, it is actually very easy to numerically solve this recurrence equation ourselves, just by using a scientific calculator!

First, input the numerical value of θ_0 and press equals $(=)$. Then, inputting the following expression and pressing equals n times (substituting Ω and K with their numerical values) returns the value of θ_n plus some integer (since we do not run the modulo operation):

$$\text{Ans} + \Omega - \frac{K}{2\pi}\sin(2\pi\,\text{Ans})$$

Then, we can simply look at the decimal part (or one minus the decimal part for negative values) to carry out the modulo operation and obtain θ_n.

B.3 Determine the periodic cycles of the map, i.e. the sequence of fixed values that θ converges to after many iterations, for $\Omega = \frac{1}{8}$ and — 12 pt

(i) $K = 2.45$,
(ii) $K = 2.75$

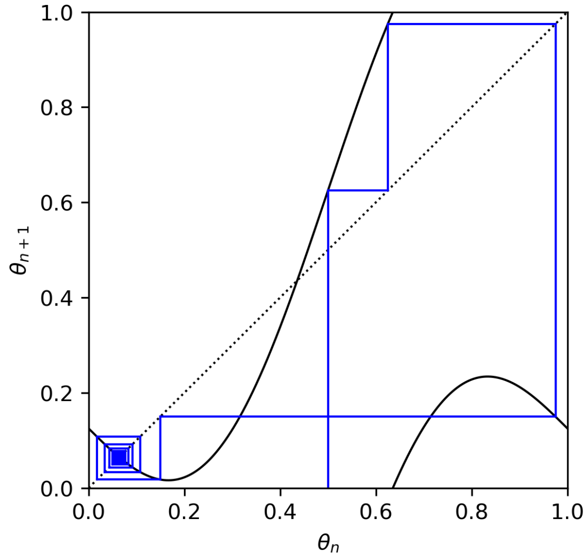
by numerically solving the recurrence equation for a sufficient number of iterations.

Leave your answer as a sequence of values to 4 significant figures, starting with the smallest value. For example, if the cycle is $0.3142 \to 0.2718 \to 0.1618 \to 0.3142 \to \cdots$*, leave your answer as* $0.1618 \to 0.3142 \to 0.2718$*.*

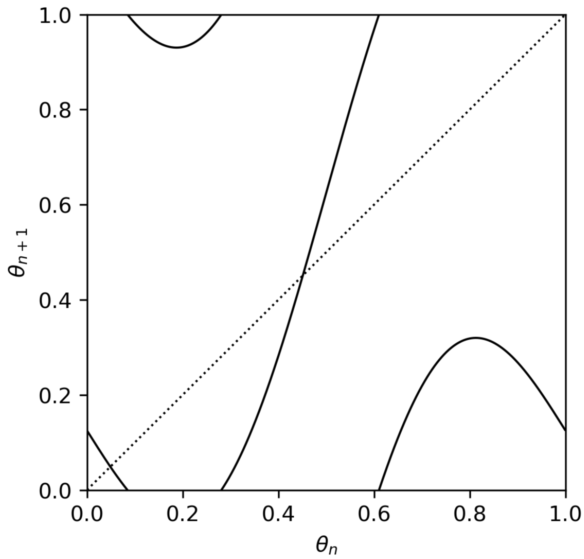
We can visualise the periodic cycles of the map using a **cobweb plot**. The cobweb plot of a map represented by the recurrence equation $x_{n+1} = f(x_n)$ consists of the diagonal line $y = x$ and a graph of $y = f(x)$. The behaviour of the map starting from some initial condition x_0 can then be visualised by following the procedure shown below:

1. Draw a vertical line from the point $(x_0, 0)$ on the x-axis to the point $(x_0, f(x_0)) = (x_0, x_1)$ on the graph of $y = f(x)$ and set $n = 0$.
2. Draw a horizontal line from the point (x_n, x_{n+1}) to the point (x_{n+1}, x_{n+1}) on the diagonal line.
3. Draw a vertical line from the point (x_{n+1}, x_{n+1}) to the point $(x_{n+1}, f(x_{n+1})) = (x_{n+1}, x_{n+2})$ on the graph of $y = f(x)$.
4. Increment n and repeat from step 2.

For example, shown below is the cobweb plot of the circle map for $K = 2$, $\Omega = \frac{1}{8}$ and the initial condition $\theta_0 = \frac{1}{2}$. Notice how the "cobweb" jumps around the plot before eventually spiralling into the stable fixed point at $\theta \approx 0.06$.



B.4 Sketch the cobweb plot of the circle map for $K = 2.6$, $\Omega = \frac{1}{8}$ and the initial condition $\theta_0 = 0.7$ on the axes provided below. 5 pt



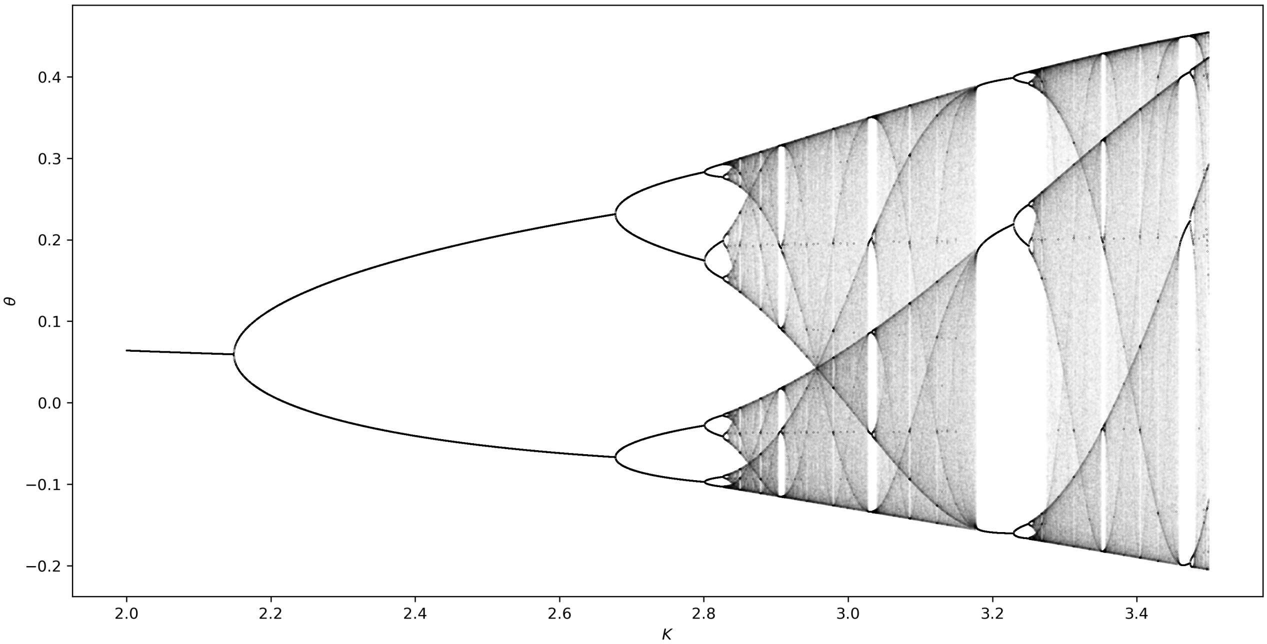
Hence, determine the period of the map for this set of parameters.

*Draw **in pencil** on the diagram in the answer sheet. Leave your answer as an integer.*

The bifurcation process can be beautifully illustrated with a **bifurcation diagram**, as shown below[1] for $\Omega = 0$. This diagram illustrates the long-term behaviour of the map as we increase the value of K, and is generated by iterating the map thousands of times for each value of K,

[1] In the diagram, the modulo operation is removed so that its typical shape is easier observed.

taking the last few hundred values of θ and plotting them on the diagram.



As we can see, we can observe the exponentially repeating bifurcations of the map starting from $\theta \approx 2.15$ and ending at $\theta \approx 2.84$, after which the map devolves into chaos and exhibits aperiodicity. Furthermore, there are actually **periodic windows**, such as the one from $\theta \approx 3.18$ to $\theta \approx 3.26$, where the map actually *regains periodicity* and bifurcates all over again!

This phenomenon of bifurcation turns out to be particularly interesting, as it demonstrates the notion of *universality*: this period-doubling property is present in most (reasonable) maps! In addition, the exponential rate of the period-doubling process can also be described by a universal constant for all maps with a second-order maximum, the (first) **Feigenbaum constant**:

$$\delta = \lim_{n \to \infty} \frac{K_n - K_{n-1}}{K_{n+1} - K_n} \approx 4.669201609\ldots$$

where K_n is the value of K at the n^{th} bifurcation. This is a charming property of chaotic maps that is simply indispensable to discussions of chaotic dynamics.

Section C. I'm Sensitive! (40 points)

A classic (and in fact defining) property of a chaotic dynamical system is that it is **sensitive to initial conditions**: a small change in the initial state to a chaotic system can result in drastically larger changes in a future state. Often, this is informally attributed to the *butterfly effect*, where "the flap of a butterfly's wings in Brazil could set off a tornado in Texas".[2]

A commonly used measure of the sensitive dependence of a chaotic dynamical system to its initial conditions is the **Lyapunov exponent**, characterising the exponential rate of divergence between two similar systems with close initial conditions. Suppose we have two systems with a small initial separation $\varepsilon_0 \ll \theta_0$, i.e. their initial conditions are θ_0 and $\theta_0 + \varepsilon_0$ respectively. Then, their divergence ε_n after n iterations can approximated as

$$|\varepsilon_n| \approx |\varepsilon_0| \, e^{\lambda n}$$

where λ is the desired Lyapunov exponent.[3] It should then be clear that $\lambda < 0$ for a non-chaotic evolution of the map and $\lambda > 0$ for a chaotic evolution of the map.

This definition of the Lyapunov exponent is super useful in numerically computing its value for a one-dimensional map. If we desire to approximate λ for a given set of parameters and initial condition θ_0, the following procedure can be used:

1. Set ε_0 to be some value much smaller than θ_0. Set N to be appropriately large such that the divergence of the two systems can be considered approximately exponential for the first N iterations.
2. Starting with initial condition θ_0, numerically solve the recurrence equation for N iterations to obtain θ_N.
3. Starting with the slightly different initial condition $\theta_0' = \theta_0 + \varepsilon_0$, numerically solve the recurrence equation for N iterations to obtain θ_N'.
4. Using its definition, solve for an estimate of the Lyapunov exponent λ using the obtained values.

C.1	Suppose we carry out the procedure described above to estimate the Lyapunov exponent for a one-dimensional map. Determine an expression for the estimate of λ using the computed values. *Leave your answer in terms of* ε_0, θ_N, θ_N' *and* N.	3 pt

Using this procedure, we are now able to obtain a rough numerical estimate of the Lyapunov exponent using only a scientific calculator. Let us now consider the circle map again. This time, we shall set $\Omega = \frac{1}{6}$.

[2] This quote is adapted from the title to a talk on predictability given by Lorenz, one of the pioneers of chaos theory with his numerical simulations of weather predictions.

[3] Conventionally, the Lyapunov exponent is the *average* of this value over many different initial conditions. We shall neglect this caveat and only consider its value for one particular initial condition at a time.

C.2 Using the procedure described previously, estimate the Lyapunov exponent λ of the circle map (from the previous section) for $\Omega = \frac{1}{6}$, $\theta_0 = 0$ and **12 pt**

(i) $K = 2.8$, using $\varepsilon_0 = 10^{-4}$ and $N = 35$,
(ii) $K = 2.9$, using $\varepsilon_0 = 10^{-8}$ and $N = 65$.

Leave your answers to 2 significant figures.

With careful consideration of the divergence between two slightly different initial states and some calculus, we can prove that the Lyapunov exponent for a one-dimensional map characterised by the recurrence equation $\theta_{n+1} = f(\theta_n)$ is given by

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \ln |f'(\theta_n)|$$

where $f'(\theta)$ denotes the slope of the map and $\lim_{N\to\infty}$ denotes that the value of expression to its right approaches λ as N increases to infinity.

This form is typically most useful for computers with the convenience of being able to store as many values of θ_n as it needs, so that it can readily compute an estimate of λ. Unfortunately, this form will not be too numerically useful for us humans. However, let's momentarily consider a different, much simpler map where this definition can be useful.

C.3 Consider the *skew tent map* that maps the interval $[0, 1]$ onto itself, defined by the recurrence equation $x_{n+1} = g(x_n)$ where **15 pt**

$$g(x) = \begin{cases} \frac{x}{\alpha} & \text{for } x \le \alpha \\ \frac{1-x}{1-\alpha} & \text{for } x > \alpha \end{cases}$$

for some parameter $0 < \alpha < 1$. The slope of the map is given by its derivative with respect with x,

$$g'(x) = \begin{cases} \frac{1}{\alpha} & \text{for } x < \alpha \\ -\frac{1}{1-\alpha} & \text{for } x > \alpha \end{cases}$$

The skew tent map is special: it is *ergodic* and has *uniform invariant density*. This means that for all initial conditions, their corresponding sequences visit every point on the interval $[0, 1]$ with **equal probability**.[b] Using this information, determine the Lyapunov exponent λ of the map as a function of α.

Leave your answer in terms of α.

[b] Technically, for *almost* all initial conditions, the sequences visit *almost* every point on the interval. What "almost" means is "all but a negligible quantity", so this technicality does not actually affect our result.

Hint: In the form given, λ is essentially a *weighted average*. If every value of $x \in [0, 1]$ has equal probability of appearing in a sequence, what does the distribution of $g'(x)$ look like?

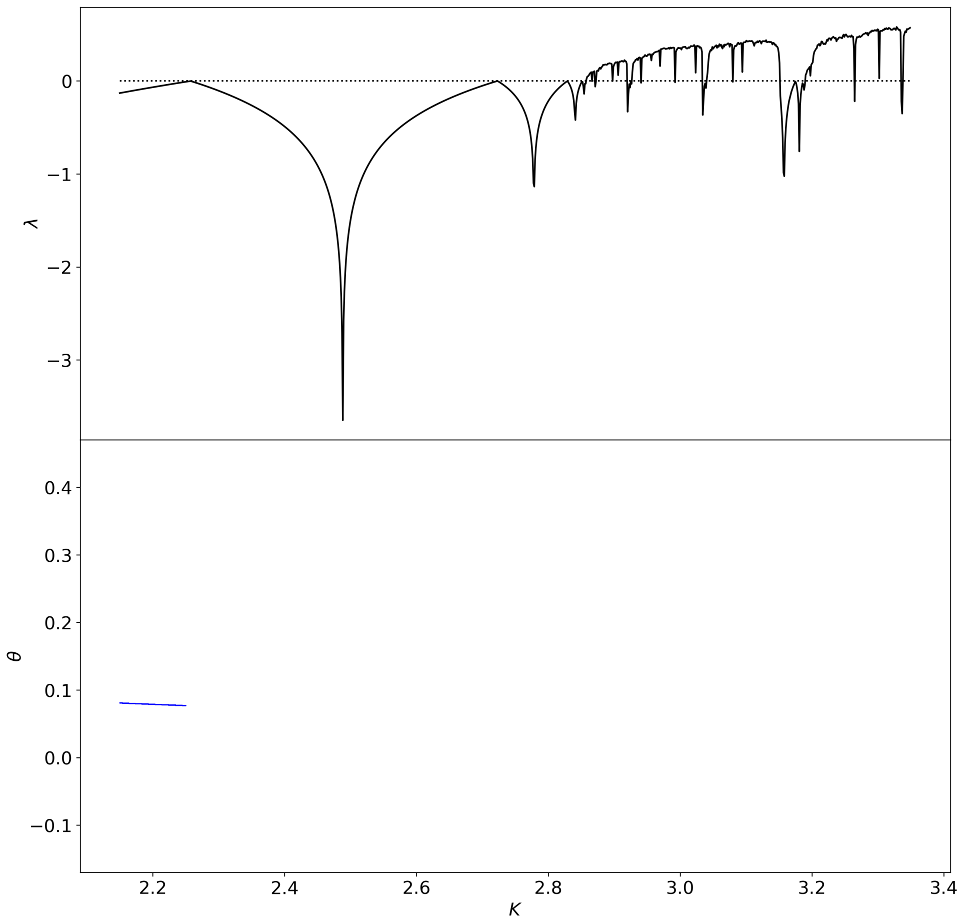
Finally, we shall present a striking connection between the Lyapunov exponent and bifurcation. Let us return to the circle map one last time with parameter $\Omega = \frac{1}{6}$.

C.4 Provided below is a plot of the Lyapunov exponent λ against the parameter K and an incomplete bifurcation diagram of the circle map with $\Omega = \frac{1}{6}$. Complete the bifurcation diagram, **indicating any important horizontal positions with respect to the plot above by drawing a dotted vertical line**. You are to demonstrate in your diagram: 10 pt

(i) the shapes displayed by the period doubling cascades in the **two** most significant non-chaotic regions,

(ii) the regions of chaos (which you may simply shade in).

No numerical values are required.



***In pencil**, complete the diagram in the answer sheet.*

Although we have not required much physical insight in this problem, chaos theory is undeniably an important interdisciplinary area of scientific study, having many different applications in science and beyond. Almost everything in the real world is governed by deterministic yet chaotic systems, which makes chaos theory integral to understanding nature.

A Shimmering Lake (100 points)

Section A. Water Waves (35 points)

Unlike light waves in vacuum, the speed of water waves depends on their wavelength. The relation between the frequency and wavelength of the wave is known as a dispersion relation. In deep water, the dispersion relation of water waves is

$$f = \sqrt{\frac{g}{2\pi\lambda}}$$

where g is the gravitational acceleration, f is the frequency of the wave, and λ its wavelength.

A.1	For a sinusoidal deep water wave with wavelength λ, derive an expression for its wave speed v. *Leave your answer in the simplest possible form in terms of g and λ.*	5 pt

The dispersion relation is different in shallow water:

$$f = \frac{\sqrt{gh}}{\lambda},$$

where g is the gravitational acceleration, h is the depth of the water, f is the frequency of the wave, and λ is its wavelength.

A.2	A wave with wavelength λ_d in deep water passes into shallow water with depth h. Derive an expression for the new wavelength of the wave after it passes into the shallow water, λ_s. *Leave your answer in the simplest possible form in terms of λ_d and h.*	10 pt

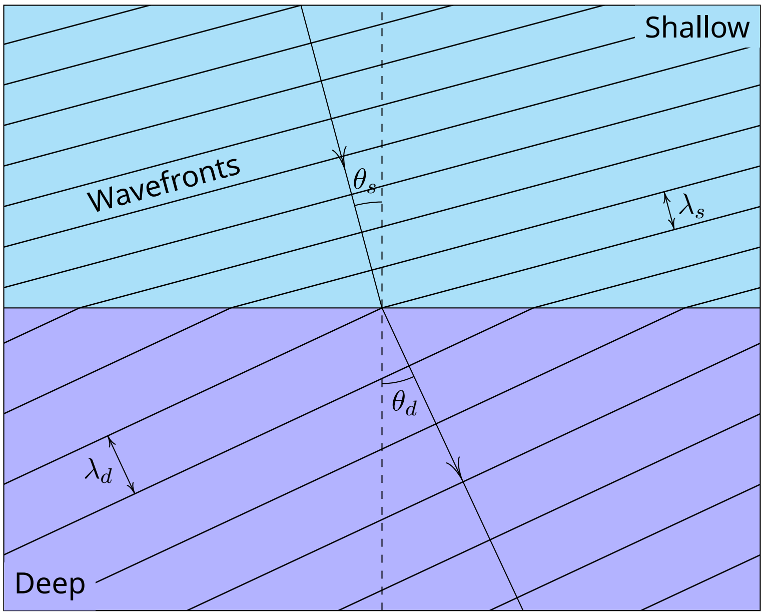
Hint: When a wave passes between two media with different wave speeds, what property of the wave remains unchanged?

You may find the diagram on the next page useful.

If the waves incident on a boundary between shallow and deep water are not parallel to the boundary, they refract according to Snell's law (which you may have seen in optics):

$$\frac{\sin\theta_s}{v_s} = \frac{\sin\theta_d}{v_d}.$$

Here, θ_s and θ_d are the angles between the direction of propagation of the waves in shallow and deep water respectively, while v_s and v_d are the wave speeds in shallow and deep water respectively.



A.3	Shallow water waves with wavelength λ_s in water with height h are incident on a boundary between shallow and deep water at an angle of incidence θ_s. Derive an expression for the angle of refraction into deep water θ_d. *Leave your answer in the simplest possible form in terms of θ_s, λ_s, and h.*	15 pt

At some critical angle θ_c, the waves will no longer pass into deep water, and will instead reflect back into shallow water. This can be a problem in ports as it can set up dangerous standing waves.

A.4	What is the critical angle θ_c for waves in shallow water entering deep water? *Leave your answer in the simplest possible form in terms of λ_s, and h.*	5 pt

Section B. Moonglade (30 points)

At night, moonglade — the reflections of the moon (and other sources of light) — can be seen on the surface of water bodies. These reflections are caused by light winds blowing across the surface of the water which cause random small-amplitude waves on the surface. The varying angle of the surface then causes multiple points to reflect light from the sources of light into our eyes, which we observe as moonglade.

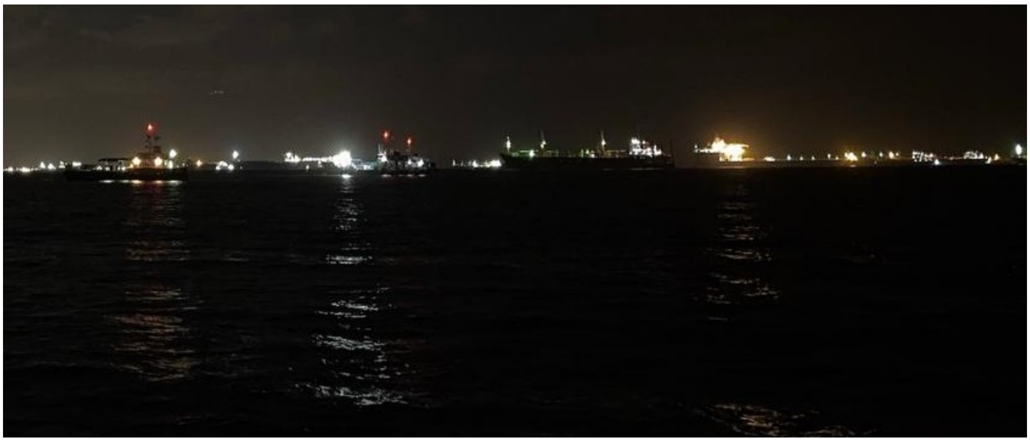
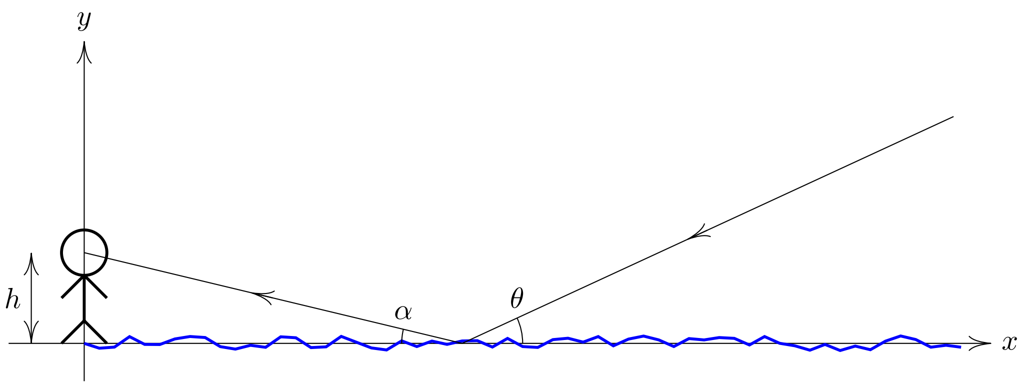


Figure 1: Reflection of light from ships seen on the surface of water

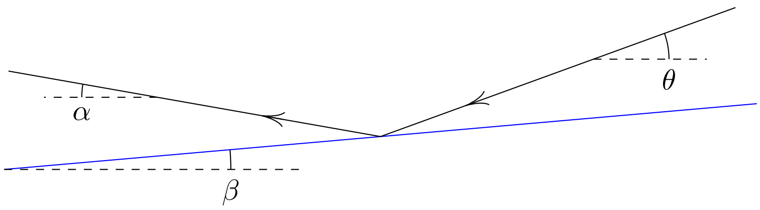
On one night, a full moon is at a small angle $\theta \ll 1$ above the horizon, and Le Xi is standing on the bank of a lake, looking in the direction of the moon. His eyes are at a height h above the surface of the lake. In this section, we will only consider waves moving directly towards the shore, so that the problem can be reduced to 2 dimensions.

We denote the angle that the ray after reflection makes with the x-axis as α. You should make the small angle approximations $\sin\theta \approx \tan\theta \approx \theta$ and $\cos\theta \approx 1 - \theta^2/2$ for $\theta \ll 1$ wherever they are appropriate.



B.1	If the surface of the lake is perfectly still, only one point on the surface reflects light from the moon into Le Xi's eyes. Determine the distance of this point from Le Xi, x_0. *Leave your answer in the simplest possible form in terms of θ and h.*	5 pt

When ripples form on the surface of the lake, they cause the surface to make an angle to the x-axis, changing the value of α. If the surface makes an angle β to the horizontal, then we can trace the reflection of light from the moon off the surface as such:



B.2	Solve for α. *Leave your answer in the simplest possible form in terms of β and θ.*	5 pt

We will consider the simple case of sinusoidal waves travelling along the x-axis toward the shore with wavelength λ and amplitude A. You may assume that $A \ll \lambda$, and that the waves do not reflect off the shore. You are given that the angle β is a function of x described by

$$\beta(x) = \arctan\left[-\frac{2\pi A}{\lambda}\sin\left(\frac{2\pi x}{\lambda}\right)\right].$$

Le Xi sees many points of light on the surface of the lake, each at a distance x_i from the shore. You may assume that $x_i \gg h$.

B.3	Determine the number of points of light which Le Xi sees on the surface of the lake, N. You may assume that $N \gg 1$. *Leave your answer in the simplest possible form in terms of θ, h, A, and λ.*	20 pt

Hint: You can solve this question graphically with a sketch — where are the solutions to $f_1(x) = f_2(x)$ located on a plot of $f_1(x)$ and $f_2(x)$ against x?

Section C. Caustics (35 points)

When sunlight shines on a shallow portion of a lake, interesting patterns of light can be observed on the floor of the lake due to the refraction of sunlight by waves on the surface of the water. These patterns are known as caustics, and will be the focus of this section.

For simplicity, we will consider the caustics formed by a series of waves with wavelength λ and amplitude A travelling along the x-axis, where $A \ll \lambda$. At an instant in time, the height of the surface is described by

$$y(x) = A\cos\left(\frac{2\pi x}{\lambda}\right).$$

This causes the surface to make an angle with the x-axis of

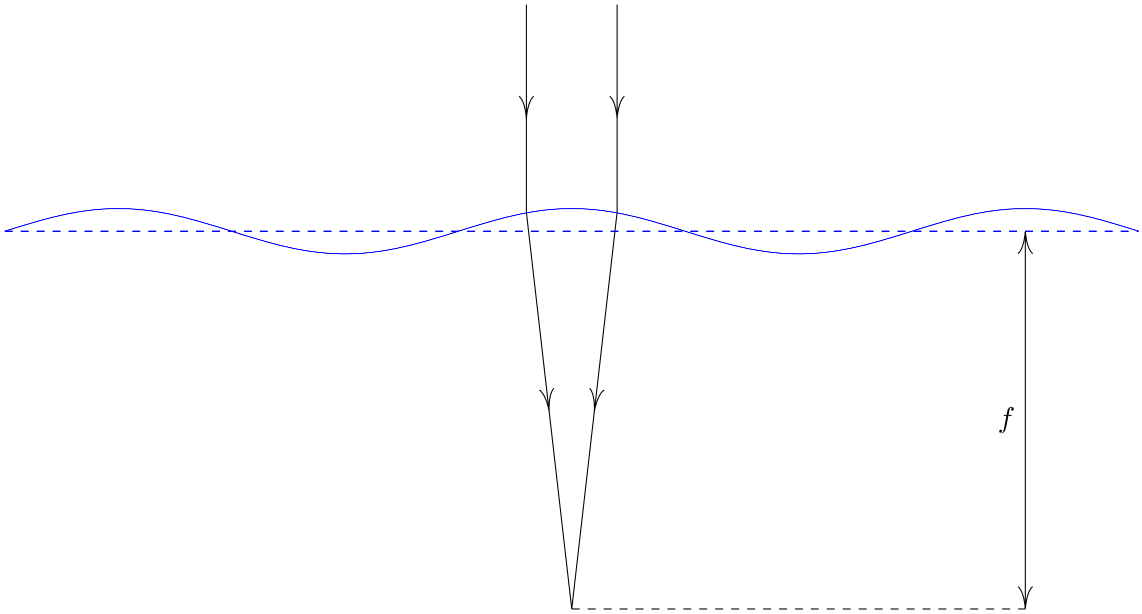
$$\beta(x) = \arctan\left[-\frac{2\pi A}{\lambda}\sin\left(\frac{2\pi x}{\lambda}\right)\right].$$

The floor is located at a height $y = -h$ such that $A \ll h$ and the water has a refractive index of n. You may assume that the sun is directly above the water such that incoming light travels downwards parallel to the y-axis. In this section, you should make the small angle approximations $\sin\theta \approx \tan\theta \approx \theta$ and $\cos\theta \approx 1 - \theta^2/2$ for $\theta \ll 1$ wherever appropriate.

C.1 If a ray of light enters the water at position x_0, at what position x will it strike the floor of the lake? 20 pt

Leave your answer in the simplest possible form in terms of x_0, n, λ, A, and h.

Light entering the water close to the peak of a wave is approximately focused to one point in space, as shown in the following diagram.



C.2	Derive an expression for the focal length of the wave, f. *Leave your answer in the simplest possible form in terms of n, λ, and A.*	15 pt

Interestingly, the patterns created by such a process (with multiple waves) turn out to be fractal-like, and have a computable fractal dimension. While this is beyond the scope of this problem, we encourage you to read up more after the competition!

Weather Woes (100 points)

Section A. Bit windy, innit? (30 points)

The atmosphere is a very complicated system to model, involving the coupling of 3-dimensional fluid mechanics and thermodynamics. However, the core principles behind many weather phenomena can be described with simple classical mechanics.

To understand weather patterns in the Earth's atmosphere, we first have to understand the length and time scales involved in the motions of its air masses. The height of the main atmospheric layer (the troposphere) is approximately $10\ \mathrm{km}$, while the horizontal extent of most weather systems can range from $1\ \mathrm{km}$ to $1000\ \mathrm{km}$. The radius of the Earth is $6378\ \mathrm{km}$. Most weather patterns evolve on timescales ranging from a few hours to a few days.

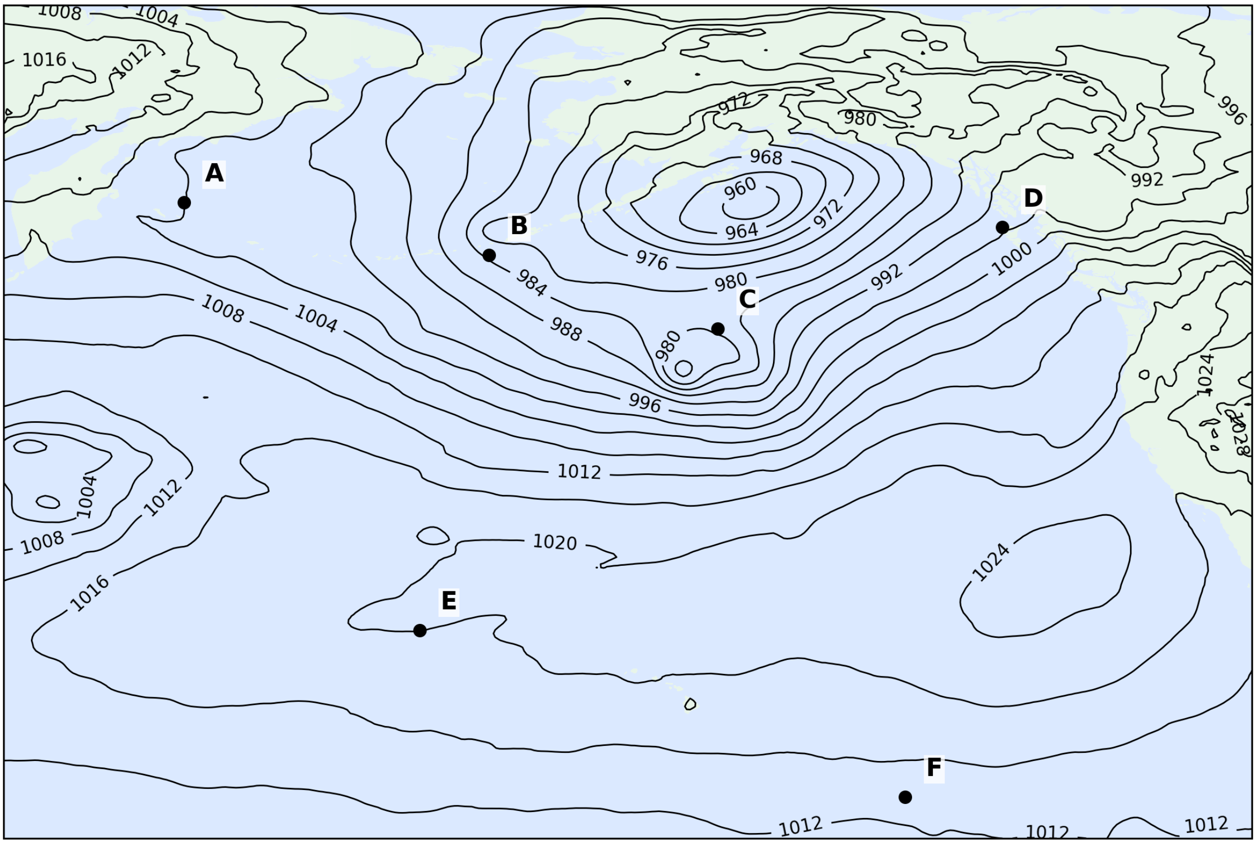
The first force we will consider is the pressure gradient force. Imagine a cube with side length l placed in a gas where the pressure increases linearly from left to right at a rate α:

$$P(x) = P_0 + \alpha x$$

A.1	Determine the net force F acting on the cube due to the pressure from the gas. F is positive if the force points to the right and negative value if it points to the left. *Leave your answer in the simplest possible form in terms of l and α.*	5 pt

In this problem, we will consider a horizontal slice of the atmosphere at constant height[1]. A contour plot showing lines of constant pressure (isobars) is shown on your answer card, where each line is labelled with its corresponding pressure value in hPa, where 1 hPa = 100 Pa.

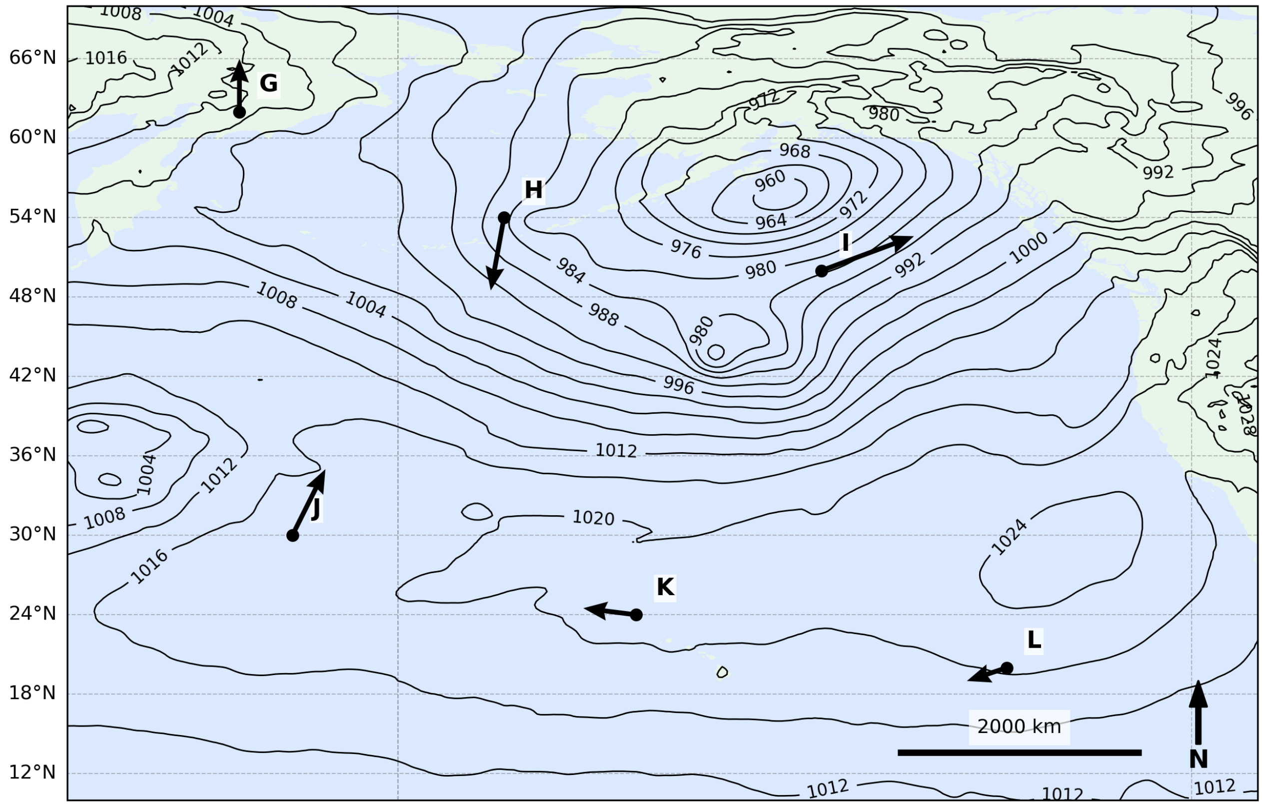
[1] Technically, this refers to constant *geopotential* height, which are surfaces of constant gravitational potential. However, we neglect the distinction in this question.



A.2	Sketch the direction of the pressure gradient force at each of the points A-F. *Draw in pencil on the diagram in the answer sheet.*	5 pt

The next force that we will consider is the Coriolis force. The Earth rotates anticlockwise about the axis passing through the north and south poles, when viewed from the north pole. Therefore, the surface of the Earth moves at different speeds at different latitudes – slower as you move towards the poles and faster as you move towards the equator. In a frame rotating with the Earth, this produces a fictitious force that acts on the air parcels as they move north or south.

The velocity vectors of air parcels at points G-L are shown on your answer card.



A.3	Find the direction of the Coriolis force acting on each of the air parcels G-L. *Draw **in pencil** on the diagram in the answer sheet.*	5 pt

The magnitude of the Coriolis force on a particle of mass m with a horizontal speed v is given by

$$F_{cor} = 2mv\Omega \sin\phi,$$

where Ω is the angular velocity of the Earth and ϕ is the latitude of the particle.

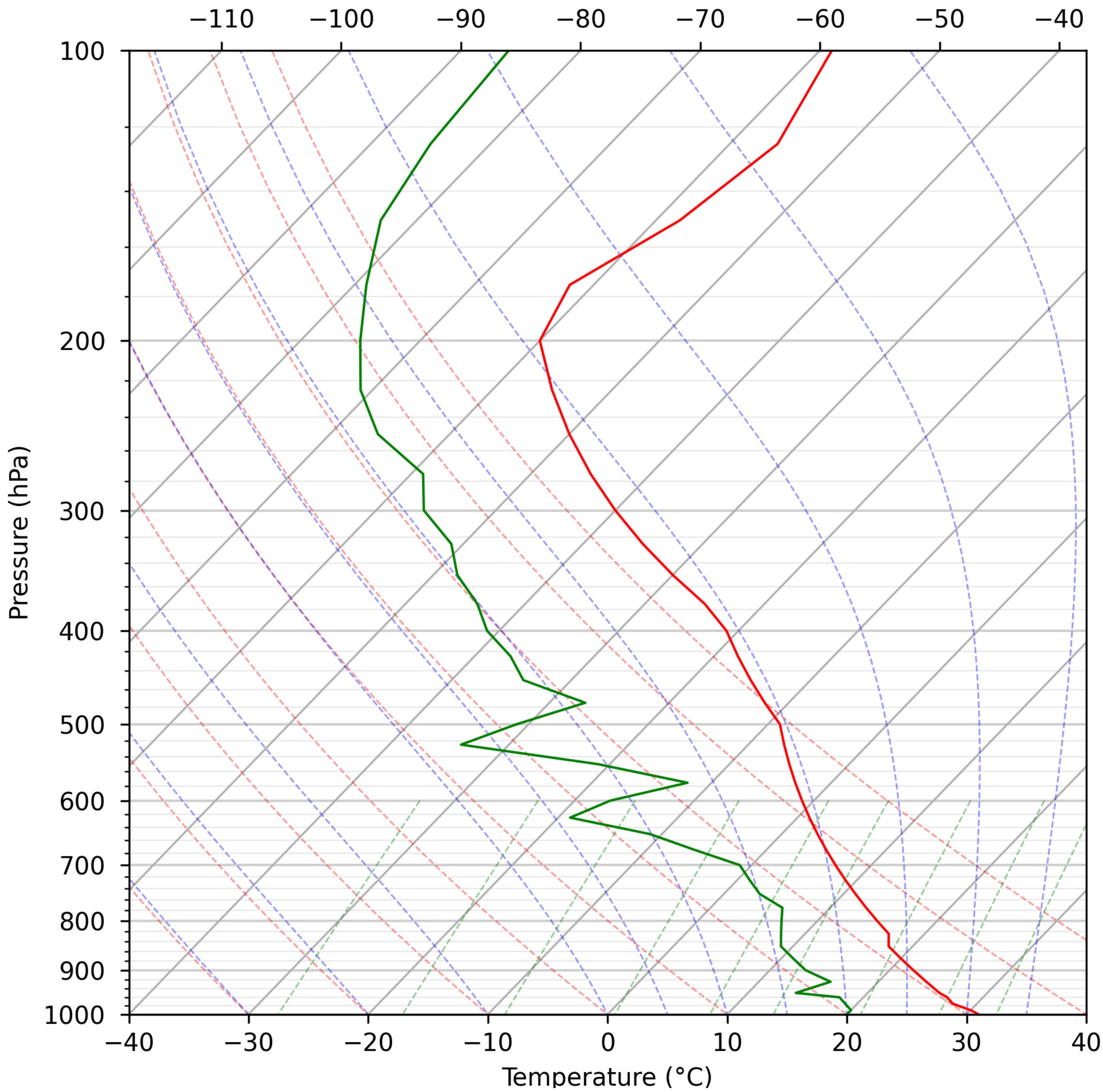
A.4	Using the length of a day, calculate the value of Ω. *Leave your answer to 3 significant figures in units of* $\mathrm{rad\ s^{-1}}$.	5 pt

In the atmosphere, the velocity vectors of air parcels eventually settle into a steady-state where the Coriolis force and the pressure gradient forces balance. In this state, the movement of air is termed **geostrophic wind**.

A.5	Using the diagram shown in **A.3**, estimate the speed of the geostrophic wind at point I. The density of air is $\rho = 1.30\ \mathrm{kg\ m^{-3}}$. *Leave your answer to 2 significant figures in units of* $\mathrm{m\ s^{-1}}$. *A relatively large margin of error is given for this part.*	10 pt

Section B. Sounds like rain (35 points)

To make accurate weather predictions, meteorologists collect data via weather balloons (radiosondes). This data is often represented on a Skew-T-Log-P (STLP) diagram, as shown below. Pressure is plotted on the y-axis in hPa, where 1 hPa = 100 Pa, and temperature is plotted on the x axis (note that the temperature axis is **skewed**; i.e, lines of equal temperature are slanted, as depicted by the diagonal thin black lines on the plot).



Let us briefly discuss how to read this graph. We model the atmosphere by considering the behaviour of small parcels of air rising and descending. As the height of the parcel increases, its pressure decreases, causing it to cool adiabatically. The theoretical temperature of parcels of air cooling in this way are represented by the dashed red lines on the plot. However, if the parcels are saturated with water vapour, the condensation of water vapour and release of latent heat reduces their cooling rate. The temperature of parcels of air cooling in this way are represented by the dashed blue lines on the plot.

The solid red line represents the **environmental temperature**, T, measured by the weather balloon at that altitude, while the solid green line represents the **dewpoint temperature**, T_d at that altitude, which is the temperature which environmental air must be cooled to for water vapour to condense into liquid water. T_d is dependent on environmental humidity and pressure.

B.1	A **temperature inversion** occurs when the temperature starts rising as height increases, instead of decreasing steadily. Identify the pressure level p_i at which a temperature inversion occurs in this graph. *Leave your answer to 3 significant figures in units of* hPa.	5 pt

B.2	Using the difference $T - T_d$, determine the pressure level p_{dry} of the driest layer between 500 and 1000 hPa. *Leave your answer to 3 significant figures in units of* hPa.	5 pt

B.3	In this diagram, clouds start forming at a pressure level 850 hPa. Hence or otherwise, on the STLP graph, draw the path of an air parcel rising from ground level to pressure level 100 hPa. *Draw in pencil on the diagram in the answer sheet.*	15 pt

The relationship between pressure P and height above ground H can be approximated as

$$H = H_0 \ln\left(\frac{P_0}{P}\right)$$

Where $H_0 = 8\,\mathrm{km}$ and $P_0 = 1000\,\mathrm{hPa}$.

B.4	Convection occurs when the rising air parcel is more buoyant than the surrounding air. Using your answer in **B.3**, determine the maximum altitude where convection is possible. *Leave your answer to 3 significant figures in units of* km.	10 pt

Section C. Hurry Hurry Hurricane (35 points)

The low pressure region depicted in the part A originated from a tropical cyclone (also called a typhoon or hurricane, depending on the region). In this section, we will investigate the airflow inside such a cyclone in more detail.

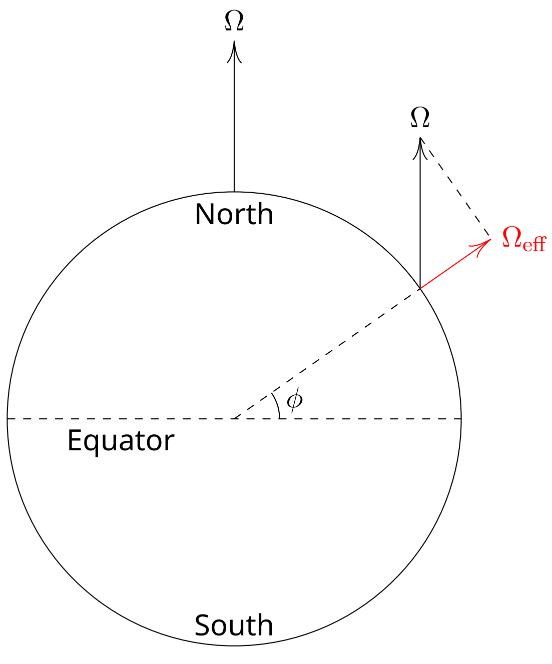
The surface pressure of tropical cyclones is lowest at their centre, and increases with distance away from the centre. The pressure profile can be approximated by

$$p(r) = p_c + (p_0 - p_c)\, e^{-R/r},$$

where p_c is the central pressure of the cyclone, p_0 is the pressure far from the cyclone and R is the characteristic length of the cyclone.

C.1	Sketch the graph of $p(r)$ against r, labelling all intercepts and asymptotes. *Draw in pencil in the space on the answer sheet.*	5 pt

Far from the cyclone, air is stationary with respect to the surface of the Earth. However, since the Earth is rotating at angular velocity Ω, this air still has angular momentum about the centre of the cyclone. The angular velocity of an air parcel about a point at latitude ϕ is equal to Ω_{eff}, the component of the Earth's angular velocity perpendicular to the surface at that point, as shown in the following diagram:



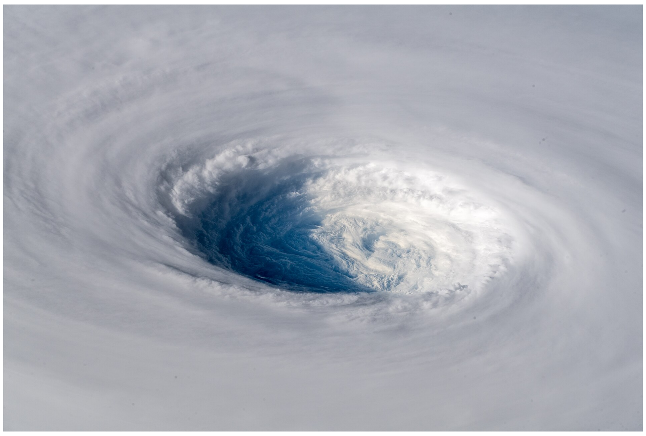
In future parts, we will ignore the friction acting on air parcels as they spiral towards the low-pressure centre of the hurricane.

C.2	An air parcel with mass m is a distance r from the centre of the cyclone, and is stationary with respect to the surface of the Earth. Determine the angular momentum L of this parcel about the centre of the cyclone. You may assume r is much smaller than the radius of the Earth. *Leave your answer in the simplest possible form in terms of m, r, ϕ, and Ω.*	5 pt

The air parcel spirals into the cyclone from an initial radius r_0 to a final radius r, gaining a tangential velocity v about the centre of the cyclone in accordance with the conservation of angular momentum.

C.3	Derive an expression for v. *Leave your answer in the simplest possible form in terms of r_0, r, ϕ, and Ω.*	5 pt

At the centre of a tropical cyclone is its eye, where winds are slow and the air pressure is extremely low. The eye is bordered by the eyewall, a ring of rapidly spinning air.



At the eyewall, the pressure gradient force keeps air parcels in uniform circular motion about the centre of the cyclone. The gradient of the pressure $\alpha(r)$ is given by

$$\alpha(r) = (p_0 - p_c)\frac{R}{r^2}e^{-R/r}$$

C.4	The eyewall of a cyclone is located a radius r_w from its centre. Determine the radius r_0 from which air had to come into the cyclone to form an eyewall at this radius. You may neglect the Coriolis force. *Leave your answer in the simplest possible form in terms of p_0, p_c, R, r_w, ϕ, and Ω.*	10 pt

Finally, we can estimate the maximum strength of a tropical cyclone by considering its thermodynamics. A mature tropical cyclone can be modelled as an ideal Carnot heat engine operating between the warm lower atmosphere and the cold upper atmosphere (the tropopause).

A cyclone moves over a warm patch of ocean where the sea-surface temperature is $T_h = 31\,°\mathrm{C}$, while the temperature near the tropopause is $T_c = -70\,°\mathrm{C}$. The hurricane extracts heat from the ocean at a rate per unit area given by

$$q = \rho C_k \Delta k v$$

While the wind drag dissipates power per unit area

$$P_d = \rho C_d v^3$$

Where $C_k = C_d = 1.5 \times 10^{-3}$ are empirical heat transfer coefficients, $\Delta k = 3.0 \times 10^4\,\mathrm{J\,kg^{-1}}$ represents the difference in heat per unit mass between ocean and air, $\rho = 1.30\,\mathrm{kg\,m^{-3}}$ is the density of air, and v is the wind speed. Assume all power dissipated by drag becomes heat in the lower atmosphere.

C.5	Compute the maximum possible wind speed, v_{max}, of the cyclone. *Leave your answer to 3 significant figures in units of* $\mathrm{m\,s^{-1}}$.	10 pt

This is termed the **Maximum Potential Intensity** of the cyclone. In reality, this intensity is approached only in very rare cases where other conditions are ideal.

T6-A
SPhL Grand Finals

Trail Mix (100 points)

Section A. Multiple Choice (25 points)

Most Physics Olympiads held internationally use long answer problems. However, as an ode to the Singaporean education system, we offer you this set of multiple choice questions, each with four options.

Solve the following multiple choice questions. You only have **three** tries for each question.

A.1 Two equal masses are connected by a spring with zero rest length, and placed on a table with a small (but nonzero) coefficient of friction. The masses are then given equal and opposite velocities perpendicular to the spring such that they move around their common centre of mass in a circle. How does the angular velocity of the rotation change with time? 5 pt

(A) The angular velocity decreases with time.
(B) The angular velocity increases with time.
(C) The angular velocity increases, then decreases.
(D) The angular velocity remains constant.

Select an option from A to D.

A.2 A neutral, spherical and solid conductor is spun around an axis at a constant angular velocity. What is the nature of the electric field within the conductor at steady state? 5 pt

(A) The electric field points radially outward from the axis.
(B) The electric field points radially inward to the axis.
(C) The electric field points radially outward from the centre.
(D) The electric field points radially inward to the centre.

Select an option from A to D.

A.3 Two people (A and B) are on an expedition to the North Pole! They decide to have fun and play a game of catch. They stand diametrically opposite from the geographic North pole (where the Earth's axis of rotation passes through the Earth's surface), facing each other. Person A throws a ball with a velocity directed towards Person B in their frame of reference. Where does the ball end up? 5 pt

(A) The ball passes to the left of B.
(B) The ball passes to the right of B.
(C) The ball overshoots B.
(D) The ball undershoots B.

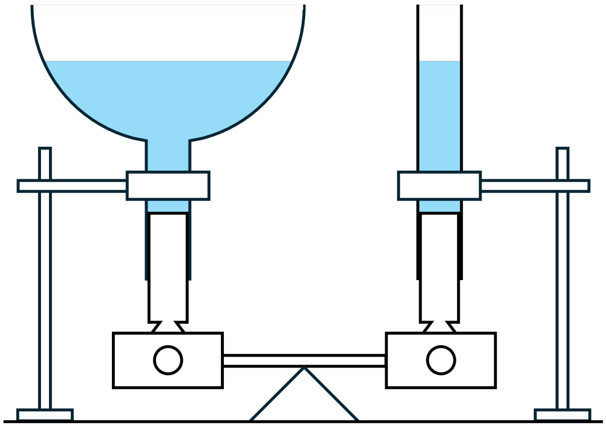
Select an option from A to D.

A.4 A rope is hung between two walls and a short transverse wave pulse is sent from the centre of the rope, outwards to the walls. How does the length and the amplitude of the pulse change with distance from the centre? 5 pt

(A) The length increases while the amplitude increases.
(B) The length increases while the amplitude decreases.
(C) The length decreases while the amplitude increases.
(D) The length decreases while the amplitude decreases.

Select an option from A to D.

A.5 In the setup below, two differently shaped tubes are filled with water to the same level and held up by a retort stand. The mouths of the tubes have the same area and each contain a piston that is able to move freely within its tube. Both pistons lie on a balancing scale that is initially level. 5 pt



The water in both tubes are then heated such that their temperatures increase by the same amount. Where does the scale tilt, and why?

(A) It tilts towards the left, since there is more water in the left funnel.
(B) It tilts towards the right, since the water height increases more in the right funnel.
(C) It remains level, since the weight remains the same on both sides.
(D) The scale could not have been level in the first place, since the normal forces on each side were initially unequal.

Select an option from A to D.

Section B. Fermi Estimation (40 points)

Fermi estimation allows us to estimate the order of magnitudes of calculations which may be tedious to do analytically. The following is an example of such a problem:

If the mass of one teaspoon of water could be converted entirely into energy in the form of heat, how many litres of water, initially at room temperature, could it bring to a boil?

The answer to this problem is $\sim 10^8\ \mathrm{L}$. This can be obtained from the knowledge of mass-energy and using common thermodynamic data of water.

Some problems are not this straightforward, as there may be data that you require but are not provided. This is where the real fun of Fermi estimation problems begins — use your intuition and common-sense to guess the order of magnitudes of these unknown quantities!

Solve as many of the ten Fermi estimation problems below as you can. You only have **three** tries for each question. The problems are not ordered by difficulty. A range of answers are given for each problem and you will be marked correct if your answer falls within the range.

Note: For all parts, submit the **base-10 logarithm of the numerical value**! The desired units will be given to you. For instance, if your answer is $2 \times 10^8\ \mathrm{L}$, write $\log_{10}(2 \times 10^8) = 8.3$ in your answer sheet.

Good luck!

B.1	At the beginning of round one, a $3\,\mathrm{cm} \times 3\,\mathrm{cm}$ cube of ice was removed from a freezer and placed on a small baking tray of area $\sim 100\,\mathrm{cm}^2$ in the Organising Team's room (which has similar conditions to the test venue). By how much did the mass of the tray and ice cube change over the course of the two-hour round? *Leave your answer to 1 decimal place as* $\log_{10}$ *of its numerical value in* kg.	4 pt

B.2	What is the power output of an average human scream? *Leave your answer to 1 decimal place as* $\log_{10}$ *of its numerical value in* W.	4 pt

B.3	What is the pressure at the centre of the Earth? *Leave your answer to 1 decimal place as* $\log_{10}$ *of its numerical value in* Pa.	4 pt

B.4	The floor and roof of the exam hall are shaded with pencil lead to make a massive capacitor. What is its capacitance? *Leave your answer to 1 decimal place as* $\log_{10}$ *of its numerical value in* F.	4 pt

B.5 How many hydrogen bombs are required to produce the energy released during the annihilation of one coughing baby – anti-coughing baby pair? 4 pt

Leave your answer to 1 decimal place as $\log_{10}$ *of its numerical value.*

B.6 How much additional energy do all the participants' brains expend in solving all eight Junior Grand Finals problems, compared to if they sat and did nothing? 4 pt

Leave your answer to 1 decimal place as $\log_{10}$ *of its numerical value in* J.

B.7 How many baked potatoes you would need to eat to obtain enough nutritional energy to create one baked potato from pure energy? 4 pt

Leave your answer to 1 decimal place as $\log_{10}$ *of its numerical value.*

B.8 4 pt

How much surface charge needs to be placed on each paw of this cat for this to happen, assuming that the picture is to scale?

Leave your answer to 1 decimal place as $\log_{10}$ *of its numerical value in* C.

Hint: The electric field strength at which electrical breakdown of air occurs is $3\,\mathrm{kV\,mm^{-1}}$.

B.9 When you have a fever, by how much does your volume increase due to thermal expansion? 4 pt

Leave your answer to 1 decimal place as $\log_{10}$ *of its numerical value in* m^3.

B.10 Imagine a crowd of astronauts suddenly appears in outer space, packed as tightly as possible. How many astronauts does it take for the crowd to immediately become a black hole? 4 pt

Leave your answer to 1 decimal place as $\log_{10}$ *of its numerical value.*

Just for fun, here is one more estimation question that is worth no points and will not be marked. However, guess the closest to the actual value and you might receive a shout-out at the end of the competition! For this question, do not take the logarithm of your answer.

B.11 What is the average score of all Junior Grand Finals teams? 0 pt

Leave your answer as a percentage.

Section C. Dimensional Analysis (35 points)

Dimensional analysis allows us to predict phenomena simply by matching the units of the relevant variables involved. Although very crude, it has its uses, as we shall see in this section. Your **data sheet** will be very useful for this subpart.

Let us first start with some standard dimensional analysis.

C.1 The **Rydberg constant** is a physical constant that relates to the spectral series of the hydrogen atom. The Rydberg constant R_H is defined as (5 pt)

$$R_H = \frac{m_e e^4}{8\varepsilon_0^2 h^3 c}$$

Determine the value of the Rydberg constant, including its units.

Leave your answer to 3 significant figures in the appropriate SI units.

C.2 Natural unit systems are measurement systems in which physical quantities are expressed in terms of combinations of fundamental physical constants. One such system is Planck units, where the speed of light c, Newton's gravitational constant G, the reduced Planck's constant $\hbar = h/(2\pi)$ and the Boltzmann constant k_B form the basis of the system. (10 pt)

Using these constants, Max Planck defined the **Planck time**, **Planck length**, **Planck mass** and the **Planck temperature**. Each of these quantities can be expressed in the form $c^\alpha G^\beta \hbar^\gamma k_B^\delta$, where $\alpha, \beta, \gamma, \delta$ are constants. By solving for these constants based on the units of the respective quantities, determine the values of the Planck time t_P, the Planck length ℓ_P, Planck mass m_P and the Planck temperature T_P.

Leave your answers as 4 expressions in the simplest possible form in terms of $c, G, \hbar$ and k_B.

Now let us see how we can apply dimensional analysis to everyday phenomena, such as in one's kitchen! Here, the problems start to require physical reasoning, as not all given variables necessarily matter in the problem.

First, we shall look at what happens when food is heated up on the stove. A large, thin slab of meat has thickness d, specific heat capacity c (heat capacity per unit mass), density ρ and thermal conductivity κ. Suppose one side (the "hot side") is suddenly heated to and maintained at some temperature T_h. A "heat front", where the temperature rises quickly, appears to slowly propagate towards the other side (the "cool side"). The cool side is initially at room temperature T_0.

C.3	Estimate the time t taken for the heat front to travel from the hot side to the cool side. *Leave your answer in the simplest possible form in terms of the appropriate variables.*	10 pt

Hint: There are more quantities given than necessary. Find relevant expressions that describe the heat flow and perform dimensional analysis on them.

A thin stream of water from a faucet collides with an upward facing spoon to form a thin, smooth sheet, as shown in the diagram below.

For this question, assume the spoon is hemispherical with radius R. We denote by v the critical velocity of the incoming water stream, above which a sheet will be formed instead of the water just filling up the spoon normally. It turns out that the values of v are different for small R and large R. Let the surface tension of water be σ, the water density be ρ and the gravitational acceleration be g.

C.4	Estimate the critical velocity v_{small} for small R and the critical velocity v_{large} for large R. *Leave your answer as two expressions in the simplest possible form in terms of the appropriate variables.*	5 pt

Hint: Not every variable is required for each expression. Use a physical argument to determine the relevant variables for each.

C.5 Estimate the critical spoon radius R at which the expression for v changes between these two critical velocities. 5 pt

Leave your answer in the simplest possible form in terms of the appropriate variables.

Although it may initially seem that dimensional analysis is the all-powerful tool in determining general dependences between variables based off purely units, we also need to know *what variables matter*. This is where physical reasoning is required, as it allows us to determine which variables actually play significant roles. Furthermore, dimensional analysis also always misses numerical prefactors, making it essentially useless for that purpose.

Nevertheless, when provided relevant variables and with some physical intuition, dimensional analysis is a very important tool. For this reason, dimensional analysis also occasionally appears even in high-level physics competitions.