6th Singapore Physics League Grand Finals (Senior)

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Problem List

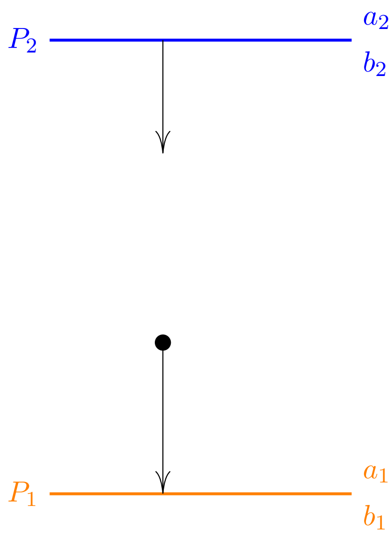
1	**Sci-fi Physics**	**1**
1A	Thinking with Portals	1
1B	Reactionless Drive	3
1C	Time Travel and Wormholes	5
2	**Really Nervous**	**1**
2A	Nerves are Like Membranes	1
2B	Nerves are Like Gates	3
2C	Nerves are Like Wires	5
3	**Just a little bit**	**1**
3A	Dualcast	1
3B	Entropic Brew	3
3C	Spinner	5
4	**Confusing Electrons**	**1**
4A	Blowin' in the Wind	1
4B	Can't Catch Me Now	3
4C	Run Boy Run	5
5	**I'm A Sucker For You**	**1**
5A	Close Enough	1
5B	Imploding Gas	3
5C	Attractive Radiation	5
6	**Spinors**	**1**
6A	Polarisation	1
6B	Spin	6
6C	Transformations	11

Sci-fi Physics (100 points)

Section A. Thinking with Portals (20 points)

Portals are a hypothetical technology that allow instant travel between distant locations. In science fiction, they usually consist of two gateways, with an object entering one leaving the other instantaneously. How would we fit this into our current model of physics?

Let us first consider a naïve model of a portal. Take each portal to be a plate with front and back sides a and b. For a connected portal pair P_1 and P_2, when one enters side a_1 of portal P_1 with velocity v, it exits from side b_2 of portal P_2 with the same velocity, and vice versa. Assume that the portals do not exert any force on objects.



A.1	Obviously, this model does not conserve energy. Consider two portals, one a distance $h = 1\,\mathrm{m}$ above the other, in a uniform gravitational field $g = 9.81\,\mathrm{m\,s^{-2}}$. A block of mass $m = 1\,\mathrm{kg}$ is released from rest, just above the bottom portal. Determine ΔE, the increase in total mechanical energy (kinetic and gravitational potential) after $t = 10\,\mathrm{s}$. *Leave your answer to 2 significant figures in units of* J.	5 pt

We can fix this model by allowing the portals to affect the gravitational field around them! Consider a **mass dipole layer**, consisting of a large layer of areal mass density (mass per unit area) σ a distance δ above a layer of negative areal mass density $-\sigma$.

A.2	Determine the magnitude of the potential difference, $\Delta\Phi$, across the dipole layer. *Leave your answer in the simplest possible form in terms of σ, δ, and any relevant physical constants.*	10 pt

Hint: This system is analogous to an electric capacitor.

The **areal dipole moment** μ is defined as

$$\mu = \sigma\delta$$

A.3	Now, we can reconcile our original setup with physics. If both portals are treated as mass dipole layers, energy can be conserved when travelling through them. Determine the effective areal dipole moment μ for this to be possible. *Leave your answer to 2 significant figures in units of* $\mathrm{kg\,m^{-1}}$.	5 pt

While this is only a toy problem, it bears quite a few similarities to "realistic" portals, or wormholes – here, negative mass is also required for energy conservation and stability. After the competition, look up Einstein-Rosen bridges for more information. :)

T1-B
SPhL Grand Finals

Section B. Reactionless Drive (35 points)

Now that we've introduced the concept of mass dipoles, let's play around with them a little. One problem with interstellar travel is the need for large amounts of fuel to act as propellant. With the use of negative mass, this constraint can be bypassed.

For all following parts, you may assume Newton's Laws hold for any mass m.

B.1	Consider a point mass $m_1 = 1.00\ \mathrm{kg}$ and negative point mass $m_2 = -1.00\ \mathrm{kg}$, separated by a distance $d = 1.00\ \mathrm{m}$ on the x-axis. After $t = 1.00\ \mathrm{day}$, determine the average displacement of the two masses $(x_1 + x_2)/2$ from their starting positions due to gravitational forces between them. *Leave your answer to 3 significant figures in units of* m.	5 pt

Now suppose you are in charge of designing a spacecraft of positive mass $M = 10000\ \mathrm{kg}$ powered with negative mass. Due to technological limitations, you only have access to a small mass $m = -100\ \mathrm{kg}$ of negative mass. You are also provided with a spring of spring constant $k = 200\ \mathrm{N\ m^{-1}}$ and rest length $l = 10\ \mathrm{m}$, which will snap if it exceeds a length of $l_{\max} = 20\ \mathrm{m}$. All components start at rest.

B.2	Determine the maximum speed v that you can accelerate the spacecraft to with the given components. You may neglect gravitational coupling between the masses. *Leave your answer to 3 significant figures in units of* $\mathrm{m\ s^{-1}}$.	10 pt

Even with an unphysical infinitely extensible spring, we are still restricted by the cosmic speed limit c. Luckily, it turns out that with sufficient negative mass, even this constraint can be broken! Consider a 1+1 dimensional spacetime with metric

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + (\mathrm{d}x - vf(x - vt)\mathrm{d}t)^2$$

where

$$f(\xi) = \exp\left(-\frac{\xi^2}{R^2}\right)$$

In general relativity, energy density is related to the spacetime metric via the Einstein field equations

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

You may use the following result without proof:

$$\rho = T_{00} = -\frac{c^2 v^2}{8\pi G}\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^2$$

Where ρ is the local mass-energy density.

B.3	Compute the total mass M required to construct this spacetime. You may find the following integral helpful: $$\int_0^\infty x^2 e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{4}$$ *Leave your answer in the simplest possible form in terms of v, R, and any relevant physical constants.*	10 pt

This spacetime acts like a "bubble" moving at speed v – objects at its centre ($\xi = 0$) move with it at 0 proper acceleration. However, unlike a moving particle, there are no limits on v – it can be as high as one desires.

B.4	Imagine a spacecraft at $\xi = 0$. It sends a light beam forward, measuring a speed c locally. For an observer far away from the "bubble", what is the observed speed c' of the light beam as it travels through the "bubble"? *Leave your answer in the simplest possible form as a function of x, t, R and any relevant physical constants.*	10 pt

Hint: What is the value of $\mathrm{d}s^2$ for a beam of light?

While locally the speed of light is unchanged, to the observer, it appears that things within the bubble are moving superluminally!

Section C. Time Travel and Wormholes (45 points)

Finally, let's look at a cornerstone of science fiction: time travel. Special Relativity tells us that the particles in the universe have a speed limit – c. But what if we could break this limit?

A particle travelling faster than light at speed αc, where $\alpha > 1$, is known as a **tachyon**.

C.1	Suppose such a particle has energy E. Determine its mass m as the complex number $a + bi$, where a and b are real constants to be determined. *Leave your answer in the simplest possible form in terms of α, E, and any relevant physical constants.*	5 pt

Imagine Aaron is on a spacecraft travelling away from Earth at $v = 0.6c$, while Sid is on Earth. Both start on Earth at time $t = 0$, and have devices which can send signals using beams of tachyons. At time $t = 500$ s (in Sid's frame), Sid realises that he burnt his toast. In despair, he sends a message to Aaron, who upon receiving the message, sends one back telling Sid to check his toaster before his toast burns. Remarkably, this works.

C.2	Determine the minimum α required such that in Sid's frame, Aaron's message reaches him *before* he sends his message. *Leave your answer to 3 significant figures.*	10 pt

In fact, partly imaginary (tachyonic) mass-energy fields *do* exist in the quantum world – for instance, the vacuum of the famous Higgs field is tachyonic. Unfortunately, these fields are unstable and collapse to give positive squared mass fields; thus no communication is possible with them. We can, however, approach time travel differently.

Recall our portal pair/wormhole from previously. It turns out that, with a judicious application of special relativity, we can use it to construct a time machine.

We slowly send Sid to Alpha Centauri, 4.25 ly away[1], while Aaron remains on Earth. Each has a pair of portals, denoted as A_1, A_2, S_1, and S_2. At $t = 0$, both send one of their portals (A_2 and S_2) at speed $v = 0.8c$ to the other. Assume the portals accelerate and decelerate instantaneously, and that Earth and Alpha Centauri are at rest relative to each other. When S_2 reaches Earth at time $t = t_1$, Aaron steps through S_2, appearing on Alpha Centauri at S_1, before stepping through A_2, appearing back on Earth at A_1 and time $t = t_2$ (measured in Earth's frame).

C.3	Determine $t_2 - t_1$. *Leave your answer to 3 significant figures in units of* years.	10 pt

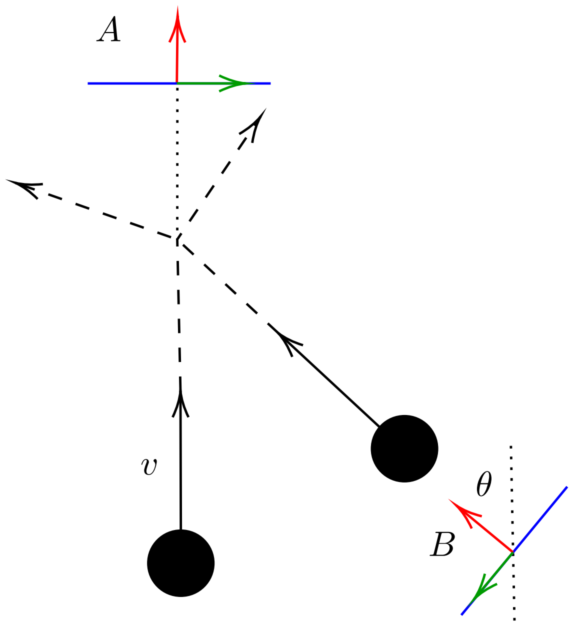
Hint: Both portal mouths correspond to the same point in space-time, and hence have the same proper time.

[1] ly are light years, the distance travelled by light in 1 year.

We could then move portal A_2 back to Earth (perhaps through portal S_2), and hence create a small local time machine.

You might be wondering whether this violates causality. Consider the following setup, proposed by **Polchinski (1990)**, with portals A and B as a completed local time machine (as described previously) with time reversal T . A billiard ball is fired at portal A with an initial velocity v, entering at time t and exiting the other end B at time $t - T$. It then collides with its younger version, preventing it from entering the portal in the first place.

Note that the second portal is rotated by an angle θ to allow for the collision to occur, and velocities and positions through the portals are conserved - i.e, a particle entering portal A at position $\mathbf{r}$ and velocity $\mathbf{v}$ exits portal B at the same corresponding position $\mathbf{r}$ and velocity $\mathbf{v}$ relative to its coordinates.



This seems pathological — however, the **Novikov self-consistency principle** states that if time travel is possible, interactions with the past should not create paradoxes. Hence, we should be able to find self-consistent solution(s) to this problem.

Omitting units for simplicity, let portals A and B be centred at $(0, 3)$ and $(1, 0)$ respectively, with B rotated by $\theta = 45°$ anticlockwise relative to A and flipped around its normal axis. The ball starts at the origin with speed $v = (0, 1)$, and the time difference between the portals is $T = \sqrt{2} + 2$. Each portal is a line segment, while the balls can be treated as point particles. All collisions are elastic.

For each self-consistent solution, let the y-position of the collision be y_c. If there exists a solution where no collision occurs, take $y_c = -10$.

C.4	Find the sum of all possible y_c. *Leave your answer to 3 significant figures.*	20 pt

Hint: A vector $\mathbf{v}$ can be rotated anticlockwise by an angle θ via a rotation matrix R_θ:

$$\mathbf{v}' = \mathbf{R}_\theta \mathbf{v} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} v_x \cos\theta - v_y \sin\theta \\ v_x \sin\theta + v_y \cos\theta \end{pmatrix}$$

Where $\mathbf{v}'$ is the rotated vector in Cartesian coordinates.

Thus: if you wish to visit your past (but not change it), first obtain a large supply of either negative or imaginary mass.

T2-A

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Really Nervous (100 points)

Section A. Nerves are Like Membranes (25 points)

Neurons operate by having a potential across their membranes. When a signal is being transmitted, the ion diffuses and the potential changes. At steady state, the potential is decided by the balance of drift and diffusion currents across the membrane.

The movement of substances under the influence of an electric field is well-modelled by the Nernst-Planck equation, which gives the solute flux of each ion J as a sum of 3 components: The diffusive flux J_{diff} (due to diffusion), the advective flux J_{adv} (due to fluid flow), and the electromigration flux J_e (due to the electric field). The 1D Nernst-Planck equation is given below.

$$\begin{aligned} J &= J_{\text{diff}} + J_{\text{adv}} + J_e \\ &= -D\frac{\mathrm{d}c}{\mathrm{d}x} + cv + \frac{Dq}{k_BT}cE \end{aligned} \tag{1}$$

Here, D is the diffusion constant **unique to each ion**, c is the concentration of the ion, q is the charge of the ion, T is the temperature, v is the fluid velocity, and E is the electric field. Also recall that the electric field is the negative of the gradient of the electric potential V, so the electromigration term may also be written as such:

$$J_e = -\frac{Dq}{k_BT}c\frac{\mathrm{d}V}{\mathrm{d}x}$$

Some data are given about the ions which may be relevant to this section of the problem. **Some entries have been intentionally omitted.**

Ion	Na^+	K^+	Cl^-
Diffusion constant D ($\mathrm{cm^2\,s^{-1}}$)	1.96×10^{-5}	1.33×10^{-5}	
Relative Channel Conductance g	0.04	1.00	0.45
Intracellular concentration c_i ($\mathrm{mmol\,dm^{-3}}$)		125	5.00
Extracellular concentration c_o ($\mathrm{mmol\,dm^{-3}}$)	145		125

The movement of these ions can be modelled using the Nernst-Planck equation. However, as given above, the equation is very challenging to solve. Hence, we will need to make some simplifications.

A.1 Select the **1** term in the Nernst-Planck equation which you believe to have the least contribution to the total solute flux across a cell membrane. 5 pt

Circle the corresponding term on the answer sheet.

For the remainder of this section, you may use the simplification made in **A.1**.

The membrane potential V_m of a neuron is defined as the electric potential of the intracellular medium relative to the extracellular medium. (i.e., the extracellular electric potential is equal to 0 by definition). This membrane potential is due to a difference in the intracellular and extracellular ion concentration.

A.2	Given that the equilibrium membrane potential for Na^+, V_{Na} is $+66.6\,\mathrm{mV}$, determine the intracellular concentration of Na^+, $c_{i,Na}$. Use a temperature of $T = 310.15\,\mathrm{K}$. *Hint:* Equilibrium is reached when there is no net flux of Na^+ through the cell membrane. *Leave your answer to 3 significant figures in units of* $\mathrm{mmol\,dm^{-3}}$.	10 pt

In reality, the total equilibrium membrane potential is $-70\,\mathrm{mV}$. This means that other ions must have contributed significantly, mainly K^+ and Cl^- (we ignore other charged compounds such as proteins because they are too large to move across membranes). We will only account for these 3 ions moving forward.

Ions may move across the membrane through certain channels, which only allow certain ions to pass through[1]. These channels can be modelled as resistors. The channels have a conductance g, defined as the reciprocal of resistance. The conductance of each type of channel relative to the K^+ channel is given in table 1.

When the membrane potential V_m is not equal to the equilibrium membrane potential of an ion (for example, Na^+), there will be ion flow (and hence current) driven by a potential difference of $(V_m - V_{\mathrm{Na}})$. When multiple ion channels are present, the channels may be modelled as resistors in parallel.

A.3	Determine an expression for the total equilibrium membrane potential V_m. *Leave your answer in the simplest form in terms of the conductances* $(g_K, g_{\mathrm{Na}}, g_{\mathrm{Cl}})$, *and the equilibrium membrane potentials of each ion* $(V_K, V_{\mathrm{Na}}, V_{Cl})$.	5 pt

A.4	Determine the extracellular concentration of K^+ $c_{o,K}$. *Leave your answer to 3 significant figures in units of* $\mathrm{mmol\,dm^{-3}}$.	5 pt

[1] For those well-versed with cell biology, we will assume that ions only move across ion-selective channel proteins. We will ignore details such as the Na/K pump.

Section B. Nerves are Like Gates (30 points)

A nerve must end at some point (for example, where it connects to another neuron). This point is known as a synapse. Here, the rise in voltage triggers the voltage-gated calcium channels to open, which results in the release of neurotransmitters into the synaptic cleft (the space between the neurons). We model the voltage-gated calcium channels as a two-state system: It may either be closed or open, and neurotransmitters are only released when it is open.

Let us assume that the state of the channels follow a Boltzmann distribution. The energy gap between the 2 states is the free energy associated with the change in conformation of the channel protein ΔG. When the membrane voltage **deviates from the equilibrium membrane voltage** by V, there is a change in energy due to the presence of the gating charge q_g, so the total energy gap will decrease by $q_g V$. Hence, the ratio of probabilities that it exists in the closed state P_{close} to the open state P_{open} is:

$$\frac{P_{\text{close}}}{P_{\text{open}}} = \exp\left(\frac{\Delta G - q_g V}{k_B T}\right)$$

B.1	Find an expression for P_{open}. *Leave your answer in the simplest possible form in terms of ΔG, q_g, V, k_B, and T*	5 pt

Each synapse contains many such channels. For simplicity, we shall assume that as long as one of the channels open, the neurotransmitters will be released. The typical voltage difference when the channel opens is $V = +40\,\text{mV}$. Some parameters are given below (e is the elementary charge):

Quantity	ΔG	q_g
Value	15 $k_B T$	$12e$

B.2	Determine the minimum number of channels such that when the voltage difference is V, the probability that the neurotransmitters are released is at least $1 - 10^{-55}$. Use a temperature of $T = 310.15\,\text{K}$. *Leave your answer as an integer.*	5 pt

If you were unable to solve **B.1**, you can use $P_{\text{open}} = 0.93$ at this voltage.

There also exists a different type of synapse which passes the signal through an electrical mechanism. We will consider a subset of this type of synapse known as a rectifying synapse. These are typically found in heart muscle, and are responsible for the synchronisation of signals. The current-voltage relationship of such a synapse was measured, and it was found that it follows the diode equation:

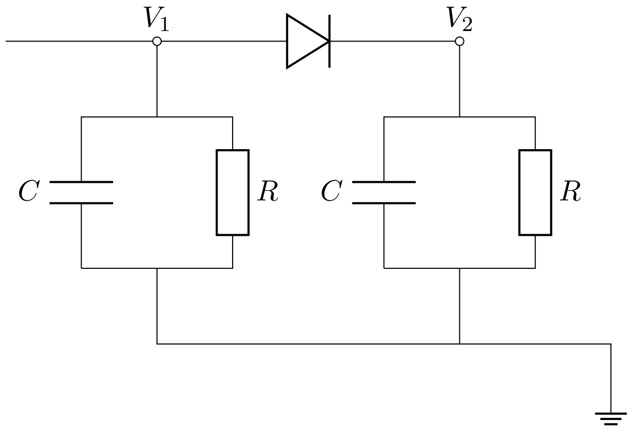
$$I = I_S\left(e^{\frac{\Delta V}{nV_T}} - 1\right)$$

Here, the current I is related to the reverse saturation current I_S, the ideality factor n, and the voltage difference across the synapse ΔV. V_T is the thermal voltage, equal to $26.726\,\mathrm{mV}$ at the physiological temperature of $T = 310.15\,\mathrm{K}$.

The parameters of the synapse are given below:

Quantity	I_S	n
Value	$3.82\,\mathrm{nA}$	0.370

The synapse will be represented by a diode in the following problems. Consider 2 cells linked up as such (each RC circuit is 1 cell):



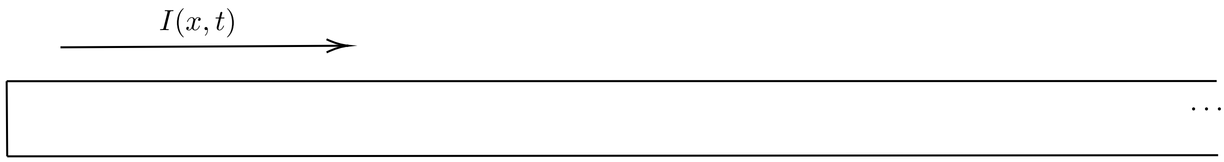
B.3	Let cell 1 be at a higher voltage than cell 2, such that the initial difference in voltages $\Delta V_0 = V_1 - V_2 \ll nV_T$. Determine an expression for the voltage difference ΔV against time. *Leave your answer in the simplest possible form in terms of R, C, I_S, n, V_T, and ΔV_0.*	10 pt

B.4	Let the cells have initial voltages $V_2 = 0$ and $V_1 \gg nV_T$. If the voltage of cell 1 is constant at V_1, then V_2 is proportional to $\ln(1 + \gamma t)$ for $t \ll RC$. Determine an expression for γ. *Leave your answer in the simplest possible form in terms of C, I_S, n, V_T, and V_1.*	10 pt

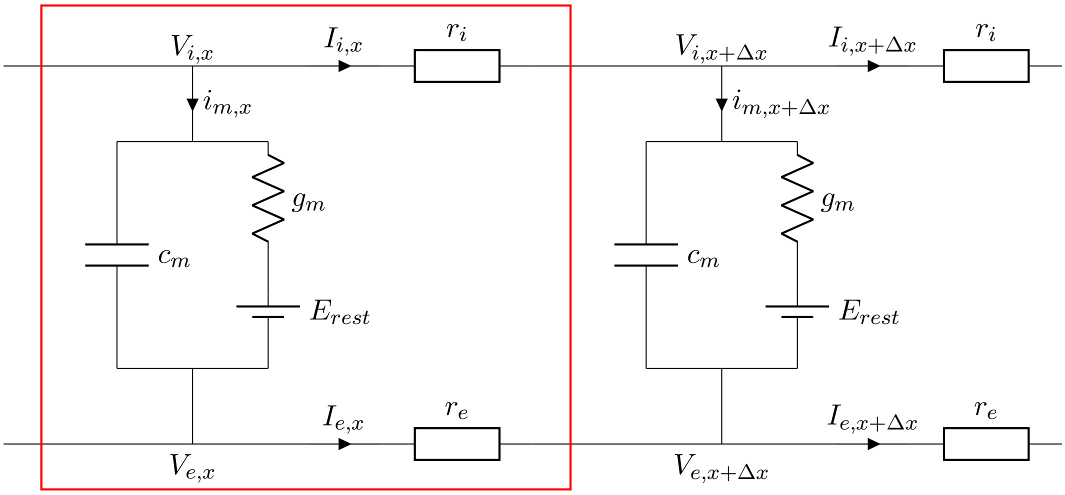
Section C. Nerves are Like Wires **(45 points)**

Some neurons can be very long, long enough that the transmission of electrical impulses can no longer be considered instantaneous because of their inductance and capacitance. Such neurons are modelled using cable theory.

Consider the following semi-infinite cylindrical neuron (has 1 end, but extends infinitely in the positive x direction):



At each point, there is some current $I(x,t)$ in the neuron. Each point also has a potential $V(x,t)$. We can model the membrane of the neuron as follows, with infinitely many repeating units of the circuit in the red box:



In the above circuit, I_i is the current flowing within the neuron, and I_e is the current in the extracellular space. i_m is the membrane current **per unit length**. r_i and r_e are the axoplasmic and extracellular resistances to current flow **per unit length**. g_m is a conductance **per unit length**, and c_m is the membrane capacitance **per unit length**. Conductance is the reciprocal of resistance. There is also a resting membrane potential E_{rest}.

Consider the section of the circuit in the red box. This part of the circuit may be considered to be "electrically small", which means that it can be assumed that the voltages and currents instantly equilibrate within the section and sections immediately adjacent to it. These sections have a length of Δx.

C.1 Determine $V_{i,x+\Delta x}$, $V_{e,x+\Delta x}$ and $I_{i,x+\Delta x}$. 5 pt

Leave your answer in the simplest possible form in terms of $V_{i,x}$, $V_{e,x}$, $I_{i,x}$, $I_{e,x}$, $i_{m,x+\Delta x}$, r_i, r_e, *and* Δx.

Most of the time, we can assume that r_e is very small, so $V_{e,x} = V_{e,x+\Delta x} = +0\,\mathrm{V}$ for all x. With this assumption, the voltage in a neuron follows a diffusion-type equation.

C.2	The voltage in a neuron can be modelled by the following partial differential equation: $$c_1 \frac{\partial V_i}{\partial t} = c_2 \frac{\partial^2 V_i}{\partial x^2} - (V_i - E_{rest})$$ By considering the limit as Δx approaches 0, determine expressions for c_1, and c_2. *Leave your answer in the simplest possible form in terms of g_m, c_m, and r_i*	10 pt

If you were unable to solve **C.2**, you may use the constants c_1 and c_2 for all subsequent problems.

C.3	If a constant current I_0 is injected into the neuron at $x = 0$, determine an expression for the steady state voltage in the neuron. *Leave your answer in the simplest possible form in terms of g_m, c_m, r_i, x, I_0, and E_{rest}.*	10 pt

Neurons can act as low pass filters for the voltage signals.

C.4	Suppose an AC voltage $V = V_0 \cos \omega t$ is applied at $x = 0$. If the angular frequency $\omega \gg \frac{g_m}{c_m}$, the steady state voltage profile is of the form: $$V - E_{rest} = V_0 e^{-k_r x} \cos(\omega t - k_i x - \theta_0)$$ By substituting the expected solution in phasor form: $V - E_{rest} = U(x)e^{i\omega t}$ into the partial differential equation in **C.2**, where i is the imaginary unit, and $U(x)$ is a spatial amplitude function, we can obtain the above expression. Hence or otherwise, determine expressions for k_r and k_i. *Leave your answer in the simplest possible form in terms of g_m, c_m, r_i, and ω*	15 pt

HINT: As a reminder, $\sqrt{i} = \frac{1+i}{\sqrt{2}}$. If you were unable to solve **C.2**, you may simply assume that ω is large.

C.5	Determine the phase velocity v_p and group velocity v_g of the voltage signal. *Leave your answer in the simplest possible form in terms of* g_m, c_m, r_i, *and* ω	5 pt

T3-A

SPhL Grand Finals

Just a little bit (100 points)

Section A. Dualcast (25 points)

You may have heard of RAM, or Random-Access Memory, in the context of computing. This is a type of computer memory that can be read or changed in any order. **Magnetoresistive Random-access Memory (MRAM)** is a kind of RAM which stores data using magnetic domains rather than electric currents and charges. In this problem, we will investigate the properties of MRAM technology.

A simplified MRAM cell consists of a layer of N bits on the $xy-$plane. The bits can be treated as identical magnetic dipoles, each of magnetic moment $\pm\mu$. Each bit can point in only 2 directions, corresponding to binary states:

Let θ denote the angle between the magnetisation vector of each bit and the positive z axis.

- Positive State (+): parallel ($\theta = 0$) to the +z, read as a value of 0.
- Negative State (-): antiparallel ($\theta = \pi$) to +z, read as a value of 1.

Suppose an external magnetic field B is applied along the $+z$ direction. The total magnetic energy of a single dipole can be written as

$$U(\theta) = A\sin^2\theta - \mu B\cos\theta$$

The first term, $A\sin^2\theta$, represents the anisotropy energy, which favours certain alignments of magnetisation along the z axis, while the second term, $-\mu B\cos\theta$, represents the potential energy of the interaction between the magnetic dipole and the external magnetic field.

Assume that the bit is initially at the state $\theta = \pi$, representing a stored value of "1". Aaron wishes to control the external magnetic field to investigate the properties of the bit.

A.1	Determine the minimum value of B Aaron must apply for the bit to undergo **spontaneous switching**, which occurs when the bit flips spontaneously from a value of 1 to a value of 0 *Leave your answer in the simplest possible form in terms of A and μ.*	10 pt

Hint: Consider a graph of $U(\theta)$. How does this change as B increases?

Aaron forces the bit to undergo a full write cycle, with the following steps:

1. From an initial bit value of 1, he starts to apply a magnetic field B and increases it slowly.
2. At some point, the bit suddenly flips to a value of 0.
3. Aaron decreases B slowly until the external magnetic field is removed.

A.2	Find the net heat Q_{slow} exchanged with the surroundings during steps 1 and 3 only. *Leave your answer in the simplest possible form in terms of A.*	5 pt

A.3 Find the total magnetic work W done on the system over an entire cycle. 10 pt

Leave your answer in the simplest possible form in terms of A.

Hint: The magnetic work W done on a bit with dipole moment $\boldsymbol{\mu}$ as the magnetic field increases from B_1 to B_2 is

$$W = -\int_{B_1}^{B_2} \boldsymbol{\mu} \cdot \mathrm{d}\mathbf{B}$$

Section B. Entropic Brew (30 points)

Often, a used MRAM cell needs to be reset, which means that the bits are randomised before new data is input.

In an MRAM cell, let N_+ and N_- represent the number of bits read as 1 and 0 respectively.

B.1	Assuming that each bit behaves independently, and that all bits are indistinguishable, determine the total number of possible microscopic dipole arrangements Ω. *Leave your answer in the simplest possible form in terms of N_+, N_- and N.*	5 pt

We define the net magnetisation of the layer of bits as $M = \mu(N_+ - N_-)$.

B.2	Determine the values of M, N_+ and N_- in the most likely configuration of the magnetic dipoles, given the external magnetic field is 0. *Leave your answers as 3 expressions in the simplest possible form in terms of N and μ.*	5 pt

The entropy of a system can be determined from the total number of possible microscopic dipole arrangements (microstates), and is given by the equation

$$S = k_B \ln \Omega$$

Where k_B is the Boltzmann constant and Ω the number of possible microscopic dipole arrangements.

B.3	Using appropriate approximations, the entropy S of the bit system can be expressed in the form $$A \ln 2 - B$$ Find A and B. *Leave your answers as 2 expressions in the simplest possible form in terms of N, k, M and μ.*	15 pt

Hint: The following approximations may be useful:

- Stirling's approximation: $\ln n! \approx n \ln n - n$
- For small x, $\ln(1 + x) \approx x - \frac{x^2}{2}$

An MRAM reset process is considered successful when thermal fluctuations dominate over magnetic alignment. In thermodynamic systems at fixed temperature, the equilibrium state is the one that minimizes the Helmholtz free energy

$$F = U - TS$$

where U is the internal energy (defined in A) and S is the entropy.

B.4	For $B < B_0$, the applied field produces extremely weak alignment of the magnetic moments. Determine B_0 *Leave your answer in the simplest possible form in terms of k, T and μ.*	5 pt

Section C. Spinner (45 points)

In a real MRAM bit, the magnetisation vector does not instantly align with the applied magnetic field. Instead, it undergoes a dynamical motion governed approximately by the Landau–Lifshitz–Gilbert (LLG) equation.

Consider a simplified MRAM bit whose magnetisation vector $\mathbf{M}$ has constant magnitude μ and components M_x, M_y and M_z. A constant external magnetic field $\mathbf{B} = B\hat{z}$ is applied.

The magnetisation dynamics are modelled by the differential equation

$$\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} = -\gamma \mathbf{M} \times \mathbf{B} - \lambda \mathbf{M_p}$$

where γ is the gyromagnetic ratio, λ is a damping constant and $\mathbf{M_p}$ is the perpendicular component of the magnetisation:

$$\mathbf{M_p} = M_x\hat{x} + M_y\hat{y}$$

Initially, the magnetisation vector points along the positive x axis:

$$\mathbf{M}(0) = \mu\hat{x}$$

C.1 The differential equations for $M_x(t)$, $M_y(t)$ and $M_z(t)$ can be expressed as — 10 pt

$$\frac{dM_x}{dt} = -aM_y - bM_x$$

$$\frac{dM_y}{dt} = aM_x - bM_y$$

$$\frac{dM_z}{dt} = c$$

Find a, b and c.

Leave your answers as 3 expressions in the simplest possible form in terms of γ, λ and B.

If you are unable to solve this subpart, you may use a, b and c in all future subparts.

C.2 Using the substitution — 10 pt

$$M_x(t) = Ae^{rt}, \quad M_y(t) = Be^{rt}$$

where A, B, and r are unknown constants, solve for $M_x(t)$ and $M_y(t)$

Leave your answers as 2 expressions in the simplest possible form in terms of μ, γ, λ, B and t.

C.3 Which of the following best describes the subsequent motion of the magnetisation vector? — 5 pt

(a) The magnetisation vector remains stationary.
(b) The magnetisation vector undergoes circular motion with constant radius but decreasing speed.
(c) The magnetisation vector spirals inward towards the z axis.
(d) The magnetisation vector spirals away from the z axis.
(e) The magnetisation vector undergoes damped oscillations back and forth in a straight line.
(f) None of the above.

Select an option from (a) to (f).

Some modern MRAMs utilise **spin-transfer torque** to change the magnetisation by passing an electric current through the device.

Electrons carry both charge and spin, which is a measure of their intrinsic angular momentum. If a current is spin-polarized, more electrons have spin angular momentum in one direction than the other. When these electrons pass through the MRAM cell, they transfer angular momentum to the bits, causing them (and their magnetisations) to rotate.

In a simplified model, the magnetisation direction is described by the unit vector

$$\mathbf{m} = \frac{\mathbf{M}}{\mu}$$

The magnetisation is initially almost aligned with the positive z axis, and can be written as

$$\mathbf{m} = m_x\hat{x} + m_y\hat{y} + \hat{z}$$

where m_x and m_y are non-zero and small.

A simplified LLG equation including spin-transfer torque can be written as

$$\frac{\mathrm{d}\mathbf{m}}{\mathrm{d}t} = -\gamma\mathbf{m}\times\mathbf{B} - \lambda\mathbf{m}_{\mathbf{p}} + \sigma\mathbf{m}\times(\mathbf{m}\times\mathbf{p})$$

Where all $\mathbf{M}$ have been replaced by their unit vector forms $\mathbf{m}$. In the above equation, σ measures the strength of the spin-transfer torque. A larger current corresponds to a larger value of σ. A spin polarized current is applied with direction

$$\mathbf{p} = \hat{z}$$

C.4 Using the vector triple product identity — 5 pt

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$$

Simplify the spin-transfer torque term.

Leave your answer in terms of σ, m_x and m_y.

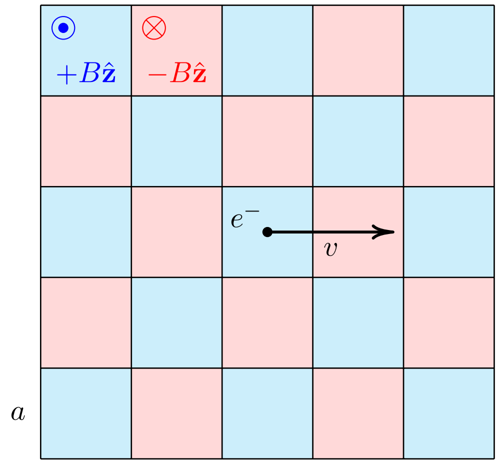
C.5 By deriving the linearised differential equations for m_x and m_y, determine the condition for σ for the magnitude of the magnetisation vector to grow rather than decay. 15 pt

Leave your answer as an expression in the simplest form in terms of λ.

Confusing Electrons (100 points)

Section A. Blowin' in the Wind (30 points)

A special magnetic field is set up which has a constant magnitude B and points along the z-axis. The direction of the magnetic field alternates in a checkerboard pattern with side length a aligned with the x and y axes, as shown in the diagram below. An electron with mass m and charge q is released from the centre of one of the checkerboard squares with speed v moving along the positive x-axis.

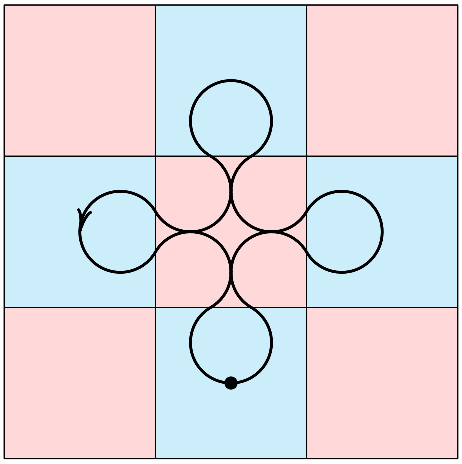


A.1	The electron moves around the checkerboard in circular arcs with radius R. Derive an expression for R. *Leave your answer in the simplest possible form in terms of* m, q, B, *and* v.	5 pt

A.2	Above a certain initial speed v_1, the electron moves along the x-axis indefinitely and monotonically — i.e. its x-velocity is always positive. Derive an expression for v_1. *Leave your answer in exact form in terms of* m, q, B, *and* a.	10 pt

Hint: Above v_1, the electron's path is periodic along the x-axis.

For some values of v, the electron moves in a closed orbit (i.e. it returns to its starting location with the same velocity it started with). One such closed orbit is shown in the following diagram.



Of all possible closed orbits, the orbit with the greatest value of $v = v_2$ passes through 8 "tiles" of the checkerboard.

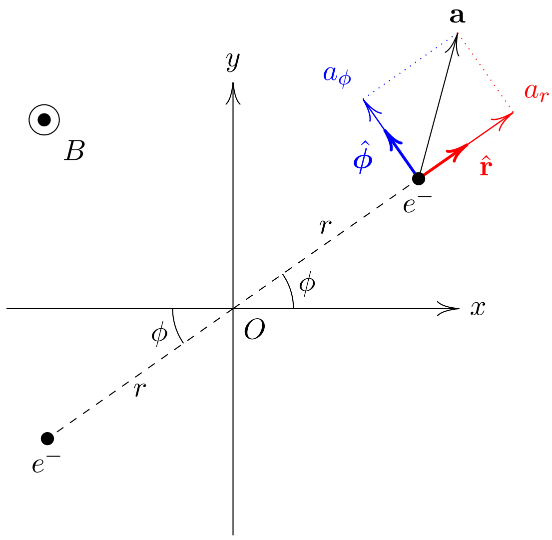
A.3	Sketch the shape of this orbit. *Draw in pencil in the space on the answer sheet.*	5 pt

A.4	Derive an expression for v_2. If you were unable to sketch the correct orbit of the electron in **A.3**, you may find the value of v corresponding to the closed orbit in the diagram above and submit that as v_2 instead. *Leave your answer in exact form in terms of m, q, B, and a.*	10 pt

Section B. Can't Catch Me Now **(30 points)**

2 electrons, each with mass m and charge q, are released from rest in a uniform external magnetic field B pointing along the $+z$-axis. The electrons are initially a distance $2r_0$ apart on the xy-plane. The origin is located at the midpoint of the line connecting the 2 electrons.

We will model this system using the polar coordinates r and ϕ, shown on the diagram below. Due to symmetry, both electrons will be at equal and opposite points (relative to the origin) throughout their motion.



The velocity of a particle in polar coordinates can be decomposed into its radial (r) and tangential (ϕ) components via $\boldsymbol{v} = v_r\hat{\boldsymbol{r}} + v_\phi\hat{\boldsymbol{\phi}}$, where

$$v_r = \dot{r},$$
$$v_\phi = r\dot{\phi}.$$

Here, a dot signifies a time derivative; i.e.

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}, \quad \ddot{x} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}.$$

B.1	By considering the magnetic and electrostatic forces acting on the electrons, derive expressions for the acceleration of one electron in the radial and tangential directions, a_r and a_ϕ respectively. *Leave your answers as 2 expressions in the simplest possible form in terms of m, q, B, r, $\dot{r}$, $\dot{\phi}$, and any relevant constants.*	5 pt

In polar coordinates, the radial and tangential accelerations of a particle are given by

$$a_r = \ddot{r} - r\dot{\phi}^2,$$
$$a_\phi = r\ddot{\phi} + 2\dot{r}\dot{\phi}.$$

Many differential equations (such as those describing the motion of the electrons) cannot be directly integrated, and we must find methods to modify them into an integrable form. One such method is through the use of integrating factors, where both sides of a differential equation are multiplied by a common factor such that they become total time derivatives.

For example, to integrate the equation

$$\ddot{x} = x,$$

we can multiply both sides by $\dot{x}$ (the integrating factor) to arrive at

$$\ddot{x}\dot{x} = x\dot{x},$$
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{x}^2\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(x^2\right).$$

Integrating both sides with respect to time (and multiplying both sides by 2) then gives

$$\dot{x}^2 = x^2 + C,$$

for some arbitrary constant C.

B.2 The equation of motion for a_ϕ can be integrated once with respect to time to obtain an expression for $\dot{\phi}$ in terms of r and its time derivatives only. Derive this expression for $\dot{\phi}$. 10 pt

Leave your answer in the simplest possible form in terms of m, q, B, r, r_0, *and any relevant constants.*

Hint: Consider the rate of change of the angular momentum of one electron about the origin, when expressed in polar coordinates.

During their motion, the electrons reach a maximum distance from the origin $r_{\max}$.

B.3 For small external magnetic fields B, $r_{\max}$ can be approximately expressed as 15 pt

$$r_{\max} = \frac{\alpha}{B},$$

where α is a constant that does not depend on B. Derive an expression for α.

Leave your answer in the simplest possible form in terms of m, r_0, *and any relevant constants.*

Hint: For small external magnetic fields, do you expect $r_{\max}$ to be very large or very small? Use this to simplify your equations.

Section C. Run Boy Run **(40 points)**

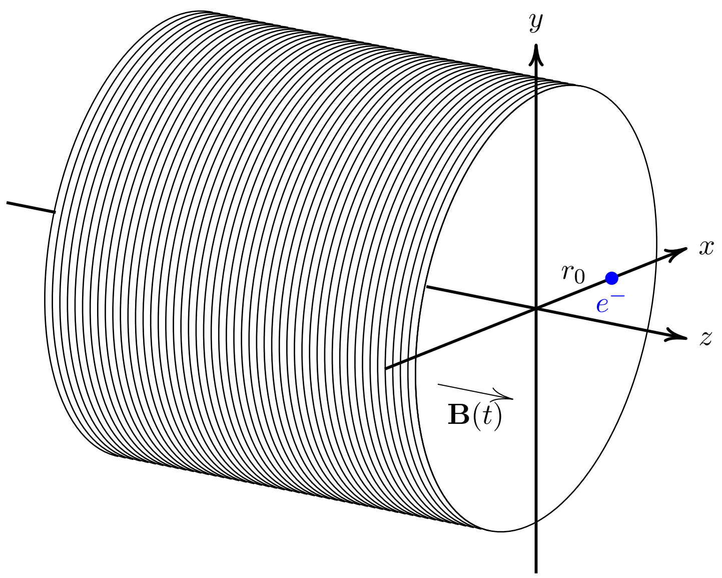
An alternating current runs through a cylindrical solenoid such that the magnetic field inside the solenoid oscillates sinusoidally in time,

$$\mathbf{B} = B_0 \cos \omega t \, \hat{\mathbf{z}}.$$

The changing magnetic field induces an electric field according to Faraday's law:

$$\oint_C \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = -\frac{\mathrm{d}}{\mathrm{d}t} \iint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{A},$$

where the integral is taken around any closed loop C and S is a surface bounded by C.



C.1	Using Faraday's law, derive an expression for the induced electric field vector (inside the solenoid) in Cartesian coordinates as a function of x, y and t. *Leave your answers as 2 expressions in the simplest possible form in terms of B_0, ω, x, y, and t. Express the vectorial components of the electric field using the unit vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$.*	5 pt

An electron with mass m and charge q is placed a distance r_0 from the centre of the solenoid along the x-axis and released from rest. We define the cyclotron frequency ω_0 of the electron as

$$\omega_0 = \frac{qB_0}{m}.$$

It turns out that this system can be solved through a complex variable substitution. We define the variable $u = x + iy$, where i is the imaginary unit, and x and y are Cartesian coordinates of the electron. The two Cartesian equations of motion of the electron can be converted to a single equation of motion in terms of u — the real component of this equation represents motion along the x-axis, and the imaginary component represents motion along the y-axis.

C.2 Derive the equation of motion of the electron in terms of u and its time derivatives. Write your equation in the form (10 pt)

$$\ddot{u} = a(u, \dot{u}, \omega_0, \omega, t),$$

where a is some function.

Leave your answer in the simplest possible form in terms of ω_0, ω, t, u, $\dot{u}$.

If we make the substitution $u(t) = f(t)v(t)$ for some complex functions $f(t)$ and $v(t)$, we can reduce this equation to the form

$$\ddot{v} + g(t)v = 0,$$

where $g(t)$ is a periodic function in time.

C.3 Derive expressions for $f(t)$ and $g(t)$. The expression for $f(t)$ contains an arbitrary constant, any value of which (within reason) will be accepted. (15 pt)

Leave your answer in the simplest possible form in terms of ω_0, ω, *and* t.

Hint: Set the coefficient of $\dot{v}$ to 0 — this should allow you to find an expression for $f(t)$.

At high values of ω, the electron travels around the origin in an approximately closed orbit.

C.4 Find the minimum and maximum distances from the origin, $r_{\min}$ and $r_{\max}$ respectively, that the electron reaches during the orbit. (10 pt)

Leave your answers as 2 expressions in the simplest possible form in terms of r_0

Hint: Does $g(t)$ tend to an "average" value at high ω? The initial values of v and $\dot{v}$ can have both real and imaginary components.

I'm A Sucker For You (100 points)

Section A. Close Enough (30 points)

The equations that we find in astrophysics are often very difficult (or impossible!) to solve analytically, and it is common to derive approximate expressions for quantities through dimensional analysis. These expressions are usually correct up to a constant factor, and most constant factors are usually "close enough" to 1, so the resultant error is tolerable for most applications (considering how difficult it is to measure anything accurately for objects which are millions of light-years away).

An example of an astrophysical equation that can be derived through dimensional analysis is Kepler's third law, which relates the period of a planet's orbit to its orbital radius and the mass of the star about which it is orbiting. We will denote the orbital radius as R and the mass of the star as M. The only other quantity which might be important for this question is the gravitational constant G, since the masses are being attracted to each other by the gravitational force. We will try to find an expression for the period T in terms of these quantities, correct up to a dimensionless constant.

The (SI) units of these quantities are

$$[R] = \mathrm{m}, \quad [M] = \mathrm{kg}, \quad [G] = \mathrm{N\ m^2\ kg^{-2}} = \mathrm{kg^{-1}\ m^3\ s^{-2}}, \quad [T] = \mathrm{s}.$$

Writing the desired equation as

$$T \propto R^{\alpha} M^{\beta} G^{\gamma},$$

we obtain the linear equations for each base SI unit

$$\begin{aligned} \mathrm{kg}: &\quad 0 = \beta - \gamma, \\ \mathrm{m}: &\quad 0 = \alpha + 3\gamma, \\ \mathrm{s}: &\quad 1 = -2\gamma. \end{aligned}$$

Solving, we find that

$$\alpha = \frac{3}{2}, \quad \beta = -\frac{1}{2}, \quad \gamma = -\frac{1}{2}.$$

Thus, we can write Kepler's third law as

$$T \propto R^{3/2} M^{-1/2} G^{-1/2}.$$

This agrees with its actual expression of

$$T = \frac{2\pi R^{3/2}}{\sqrt{GM}},$$

once again up to the dimensionless factor of 2π.

For all subparts, express your answer X in the form

$$X \propto A^a B^b C^c ...$$

where A, B, C are the relevant physical parameters and the exponents a, b, c are rational numbers.

A.1	A star has uniform density ρ and pulsates with a period T. Derive an expression for T. *Leave your answer in terms of ρ and any relevant physical constants.*	5 pt

A.2	A black hole with mass M has an accretion disk around it. Mass from the disk is constantly falling into the black hole at a rate of μ (with units of $\mathrm{kg\ s^{-1}}$). The accretion disk radiates away the energy lost by the infalling mass and has a steady-state temperature T. Derive an expression for T. *Leave your answer in terms of M, μ and any relevant physical constants.*	10 pt

Hint: Which physical constants are closely related to black holes and thermal radiation? You will also need to consider some physics — which 2 quantities always appear multiplied together when gravitation about a central body is involved?

A.3	A neutron star with moment of inertia I and radius R rotates about its axis with period T. The neutron star emits a magnetic field with magnitude B, which radiates away energy as the neutron star rotates due to relativistic effects. This causes its period to increase at a rate τ. Experimentally, it is observed that $\tau \propto T^{-1}$. Derive an expression for τ. *Leave your answer in terms of I, R, T, B, and any relevant physical constants.*	15 pt

Hint: τ is dimensionless, so you will need to think of some physical equation(s) relating it to the other variables. 2 physical constants are also required, related to magnetism and relativity respectively — good luck!

Section B. Imploding Gas (35 points)

In the initial stages of star formation, a cloud of dust and gas collapses under its own gravity to form a condensed stellar nucleus. Consider an initially static spherical cloud of gas with uniform density ρ_0 and initial radius R_0 — you may assume that in the initial state, there is no matter outside R_0.

We will assume that the effect of pressure in the gas is negligible compared to its self-gravity, so that we can model individual gas and dust particles as undergoing gravitational free-fall. You may assume that particles on the outer surface of the cloud never "overtake" those inside the cloud during the collapse, so that they always remain on the outer surface of the cloud.

For this section, we will use the convention that positive quantities point radially outwards. Be careful of the signs throughout this section!

B.1	At some time during the collapse, the cloud has an outer radius $R(t)$. Derive an expression for the the radial acceleration of particles at the outer surface of the cloud at this time, $a(t)$. *Leave your answer in the simplest possible form in terms of* ρ_0, R_0, $R(t)$, *and any relevant physical constants.*	5 pt

The expression for $a(t)$ can be integrated to obtain an expression for the velocity of particles at the outer surface of the cloud, $v(t)$, as a function of $R(t)$. This integration can be done through multiplying both sides of the equality by $v(t)$, and noting that $a(t) = \mathrm{d}v/\mathrm{d}t$ and $v(t) = \mathrm{d}R/\mathrm{d}t$.

B.2	Obtain an expression for $v(t)$. *Leave your answer in the simplest possible form in terms of* ρ_0, R_0, $R(t)$, *and any relevant physical constants.*	5 pt

This expression can then be integrated once more to obtain the free-fall collapse time of the cloud — the time taken for $R(t)$ to decrease from R_0 to 0. To perform this integration, you may find the substitution $R = R_0 \cos^2\theta$ useful.

B.3	Obtain an expression for the free-fall collapse time of the cloud, t_{ff}. *Leave your answer in the simplest possible form in terms of* ρ_0 *and any relevant physical constants.*	10 pt

The derivation above implies that any gas cloud will eventually undergo gravitational collapse. However, it turns out that only gas clouds above a certain critical radius, known as the **Jeans radius**, can begin the process of gravitational collapse. This critical radius can be derived from the virial theorem.

The virial theorem states that for a gravitationally bound system in equilibrium, the time-averaged kinetic and potential energies, $\langle K \rangle_t$ and $\langle U \rangle_t$ respectively, are related by the equation

$$2\langle K \rangle_t + \langle U \rangle_t = 0.$$

Here, $\langle x \rangle_t$ denotes the time-averaged value of the quantity x.

B.4	Derive the condition for gravitational collapse as an inequality involving the time-averaged kinetic and potential energies of the gas cloud. *Leave your answer in the simplest possible form in terms of* $\langle K \rangle_t$ *and* $\langle U \rangle_t$.	5 pt

For a static gas cloud with temperature T and N particles, its kinetic energy corresponds to the total internal energy of the gas,

$$K = \frac{3}{2} N k_B T,$$

where k_B is the Boltzmann constant.

The potential energy of the gas cloud corresponds to the total gravitational potential energy of its component particles.

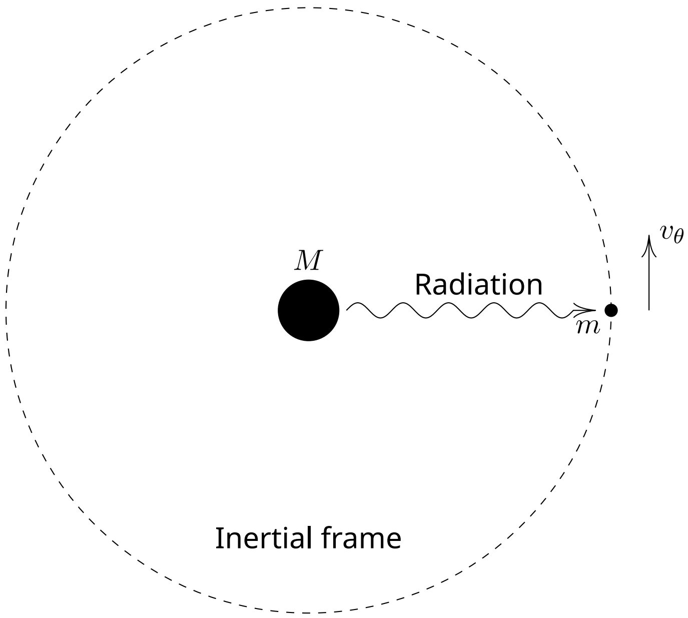
B.5	A gas cloud consisting of hydrogen atoms with mass m_H has temperature T and density ρ_0. Derive an expression for the Jeans radius of the cloud, R_J, correct up to a dimensionless constant. *Leave your answer in the form* $R_J \propto A^a B^b C^c \ldots$ *where* A, B, C *are the relevant physical parameters and the exponents* a, b, c *are rational numbers. Your answer should be in terms of* m_H, T, ρ_0, *and any relevant physical constants.*	10 pt

Section C. Attractive Radiation **(35 points)**

After a gas cloud collapses into a star, it continues to suck in unsuspecting gas particles around it. In this section, we will consider a stationary star with mass M and luminosity (total power emitted) L situated at the origin, emitting isotropically. A spherical particle of radius R and mass $m \ll M$ orbits at a distance r away.

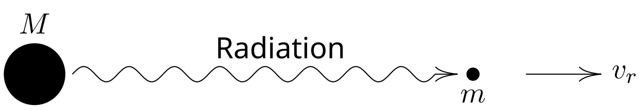
In the particle's frame, it experiences a radiation force from the star in both the radial and tangential directions. For small velocities, the radial force F_r depends only on the particle's radial velocity v_r and the tangential force F_θ depends only on the tangential velocity v_θ. Here, we use the convention that v_r is in the direction of increasing r and v_θ is anticlockwise.

Let us first find F_θ. Assume the particle is in a circular orbit around the central mass M as shown in the diagram. In the particle's frame, the radiation from the star impinges on the particle at an angle, causing it to slow down.



C.1	Find F_θ to leading order in $v_\theta/c \ll 1$. Include signs. *Leave your answer in the simplest possible form in terms of L, R, r, v_θ and c.*	10 pt

Now assume that the particle is no longer orbiting the star, but is instead moving radially away at a speed v_r as shown in the diagram below. Note that the star is still stationary.



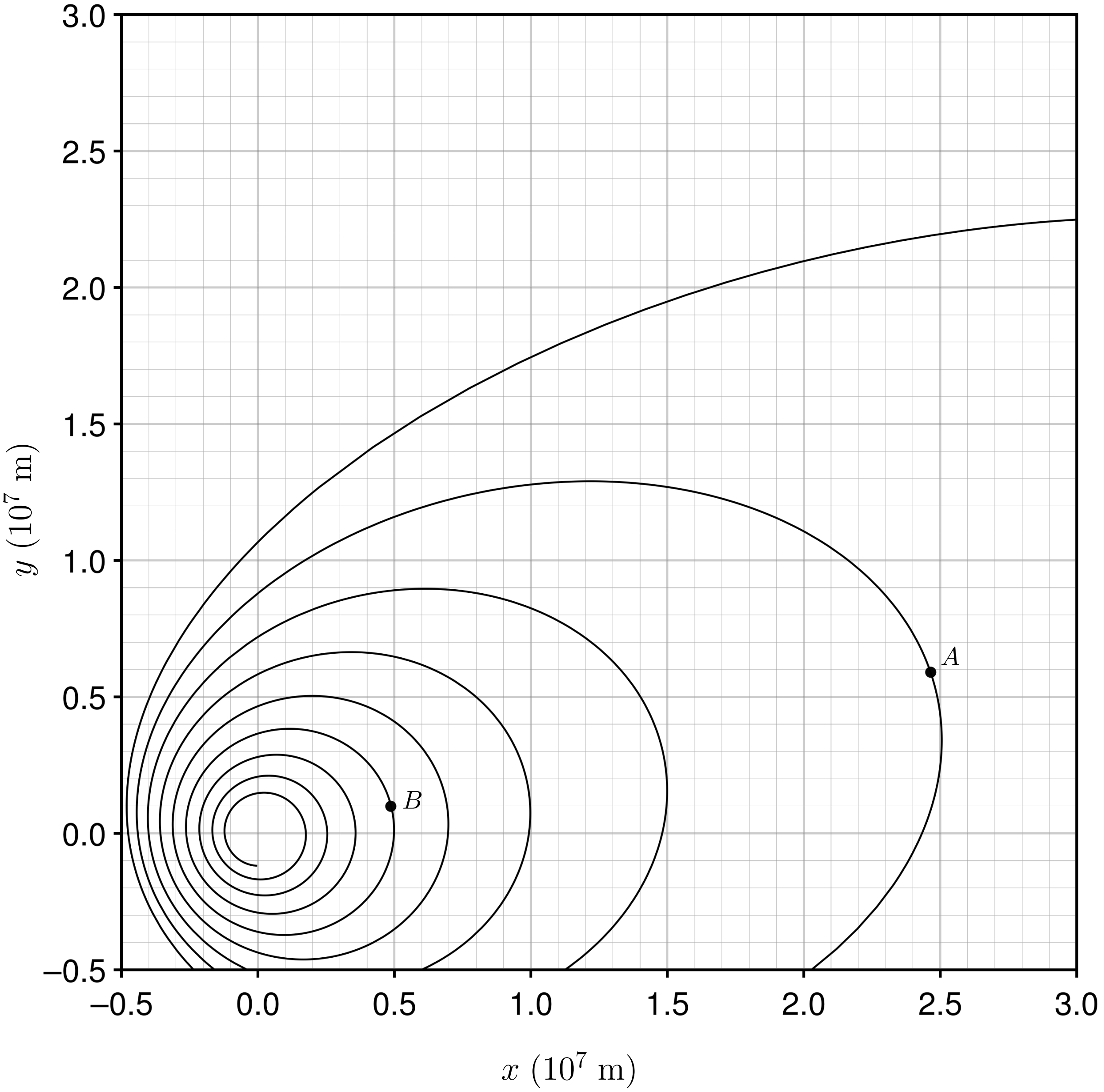
C.2	Find F_r to leading order in $v_r/c \ll 1$. Include signs. *Leave your answer in the simplest possible form in terms of* L, R, r, v_r *and* c.	10 pt

Hint: The particle's radial velocity affects both the rate of photons absorbed and the energy of each photon in its frame.

Let us consider the motion of this particle under gravity and radiation forces. On the next page you will find part of its trajectory, with the star at the origin. Two points A and B are labelled (both collinear with the origin). The position and velocity of the particle at those points are $\mathbf{r_A}$, $\mathbf{r_B}$, $\mathbf{v_A}$ and $\mathbf{v_B}$ respectively.

It is known that $\mathbf{v_A} \perp \mathbf{r_A}$, $\mathbf{v_B} \perp \mathbf{r_B}$, $|\mathbf{v_A}| = 6.70 \times 10^5\ \mathrm{m\,s^{-1}}$ and $|\mathbf{v_B}| = 2.40 \times 10^6\ \mathrm{m\,s^{-1}}$. You may also use $L = 5.0 \times 10^{27}\ \mathrm{W}$ and $m = 1\ \mathrm{g}$. The diagram is to scale and you can make measurements on it. Note that the axes of the diagram are in units of $10^7\ \mathrm{m}$.

C.3	From the diagram, determine the radius of this particle. *Leave your answer to 3 significant figures in units of* cm.	15 pt


3.0
2.5
2.0
1.5
1.0
0.5
0.0
−0.5
−0.5
0.0
0.5
1.0
1.5
2.0
2.5
3.0
y (10^7 m)
x (10^7 m)
A
B

T6-A

SPhL Grand Finals

Spinors (100 points)

Section A. Polarisation (30 points)

Mathematical background

Define the **imaginary unit** as i, satisfying $i^2 = -1$. A complex number is therefore of the form $a + bi$, where a and b are real numbers. a is known as the **real part**, while b is known as the **imaginary part**.

The **magnitude** of a complex number $z = a + bi$ is given by:

$$|z| = \sqrt{a^2 + b^2}$$

Note that the magnitude of a complex number is always nonnegative.

The **complex conjugate** of a complex number $z = a + bi$ is:

$$\overline{z} = a - bi$$

Note that $z\overline{z} = a^2 + b^2 = |z|^2$.

Complex numbers can be added, subtracted, multiplied and divided, much like real numbers. Division by complex numbers is done by multiplying both the numerator and denominator by the complex conjugate of the denominator, which allows us to make the denominator into a real number. For example,

$$(2 + 3i) + (-1 - 5i) = 1 - 2i$$

$$(1 + i)(3 - i) = 3 - i + 3i - i^2 = 4 + 2i$$

$$\frac{2 - i}{1 + i} = \frac{(2 - i)(1 - i)}{(1 + i)(1 - i)} = \frac{2 - 3i}{2} = 1 - \frac{3}{2}i$$

An important identity with complex numbers is **Euler's identity**, which holds for all real θ:

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1}$$

Matrix multiplication with vectors:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} ae + bf \\ ce + df \end{pmatrix}$$

Linear polarisation

Consider a electromagnetic wave travelling in the positive z-direction. For simplicity, we will only consider the electric field. Since electromagnetic waves are transverse, they oscillate perpendicularly to the z-axis. However, there are many possible directions perpendicular to the z-axis. The specific direction of the oscillations is known as the **polarisation** of the wave.

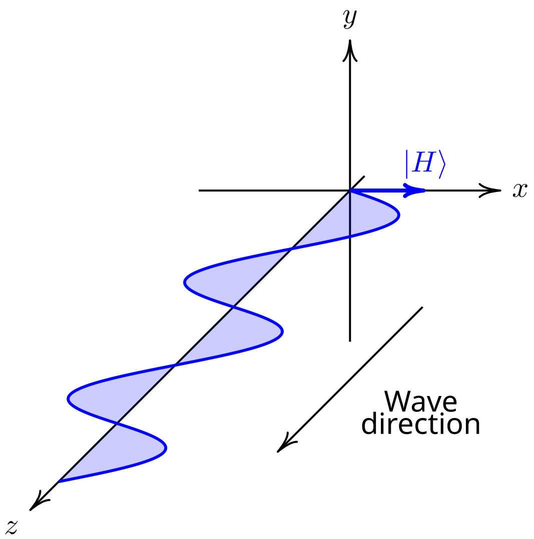
If the wave is only polarised in one direction, it is **linearly polarised**. For example, if the wave only ever oscillates in the x-direction, it is linearly polarised in the x-direction (also known as horizontal polarisation).

An arbitrary linearly polarised wave can be expressed as:

$$\mathbf{E} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} E_0 e^{i(kz-\omega t)} \tag{2}$$

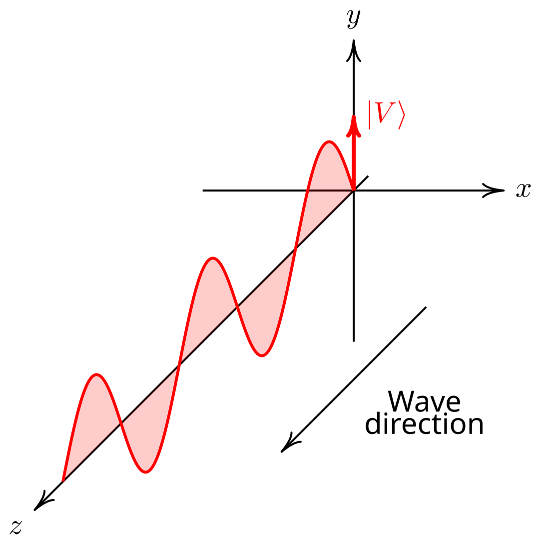
In the equation above, the term $e^{i(kz-\omega t)}$ provides the wave with its sinusoidal shape and motion. The *real part* of this term is $\cos(kz - \omega t)$ (as per Euler's identity, equation (1)), which is the physical electric field at that point – however, the use of complex numbers will make the following computations simpler.

For the other terms in equation (2), E_0 represents the amplitude of the wave, while $\binom{\alpha}{\beta}$ is a vector representing the polarisation direction. Two important examples of linearly polarised waves are shown below:



Horizontal polarisation

$$\mathbf{E} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} E_0 e^{i(kz-\omega t)}$$



Vertical polarisation

$$\mathbf{E} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} E_0 e^{i(kz-\omega t)}$$

Hence, in the vector $\binom{\alpha}{\beta}$, α represents the x-component of the polarisation and β represents the y-component of the polarisation. This vector is known as the **Jones vector**. If a Jones vector has a length of 1, i.e. $\alpha^2 + \beta^2 = 1$, it is **normalised**.

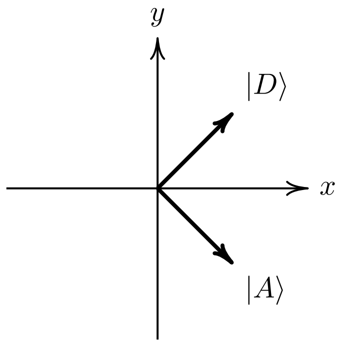
There are 2 special Jones vectors for horizontal and vertical polarisation: $|H\rangle$ and $|V\rangle$, with

$$|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad |V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Notice that every Jones vector $|J\rangle = \binom{\alpha}{\beta}$ can be written as a linear combination of $|H\rangle$ and $|V\rangle$:

$$|J\rangle = \alpha\,|H\rangle + \beta\,|V\rangle \tag{3}$$

A.1 Let the Jones vector for polarisation at an angle of 45° between the x and y axes be $|D\rangle$ ('diagonal'). Similarly, let the Jones vector for polarisation at an angle of -45° from the x-axis be $|A\rangle$ ('antidiagonal'). — 5 pt



Write down $|D\rangle$ and $|A\rangle$ as vectors of the form $\binom{\alpha}{\beta}$, **and** as linear combinations of $|H\rangle$ and $|V\rangle$. Ensure that both $|D\rangle$ and $|A\rangle$ are normalised, and that the x-components α are nonnegative.

Leave your answer as 4 expressions in the simplest possible form.

Circular polarisation

Not all light is linearly polarised. If the x and y components of the wave are not in phase, we obtain **elliptical polarisation**. Suppose the y-component lags by a phase angle of ϕ behind the x-component. Then the wave will look like:

$$\begin{aligned}\mathbf{E} &= \begin{pmatrix} aE_0 e^{i(kz-\omega t)} \\ bE_0 e^{i(kz-\omega t+\phi)} \end{pmatrix} \\ &= \begin{pmatrix} a \\ be^{i\phi} \end{pmatrix} E_0 e^{i(kz-\omega t)}\end{aligned}$$

We can generalise the Jones vector[1] $|J\rangle = \binom{\alpha}{\beta}$ by allowing it to have complex components α and β. Now, the 'length' of a Jones vector $|J\rangle = \binom{\alpha}{\beta}$ is given by

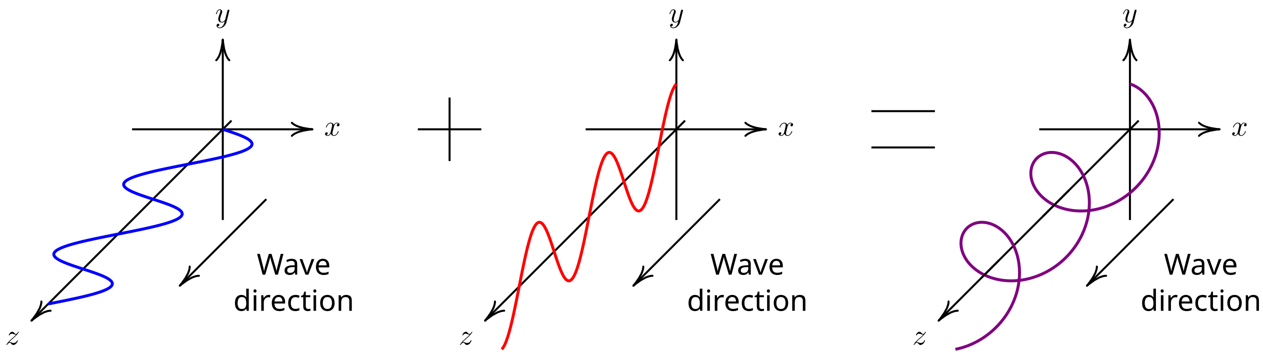
$$\sqrt{|\alpha|^2 + |\beta|^2} \tag{4}$$

[1] Note that a Jones vector is NOT necessarily an actual vector in 3D/2D space anymore; it is a mathematical object containing 2 complex numbers as its components, that behaves like a vector in an abstract '**state space**'. The name Jones 'vector' is thus a bit of a misnomer.

Note that a normalised Jones vector still needs to have a 'length' of 1.

To fully understand elliptical polarisation, lets consider a special case. Suppose that the x- and y-components of the wave are equal, but the y-component is a phase of exactly $\frac{\pi}{2}$ ahead.

When we add the 2 components together to form the resulting wave, we obtain:



From the perspective of the source of the wave, the wave is rotating circularly! As such, this is known as **right-handed circularly** polarised (RCP) light (this is because when viewed from the source looking in the $+z$ direction, the rotation of the wave and its direction of travel follow the right-hand rule). The Jones vector for this is therefore:

$$\begin{pmatrix} 1 \\ e^{-i(\pi/2)} \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

However, this is not normalised; to normalise it, we just need to multiply by $1/\sqrt{2}$:

$$|R\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

A.2 Similar to RCP light, there is also **left-handed circularly** polarised (LCP) light. Work out the Jones vector for LCP light, $|L\rangle$. By convention, ensure that the Jones vector is normalised and the x-component is a nonnegative real number. 5 pt

Leave your answer in the simplest possible form.

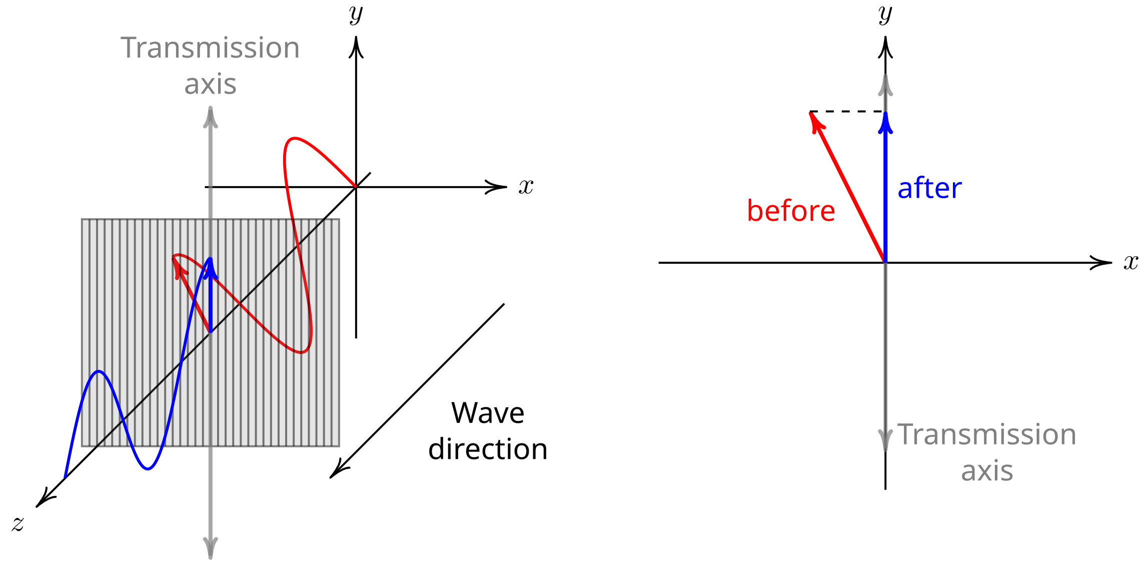
A.3 Rewrite the Jones vectors $|H\rangle$ and $|V\rangle$ as linear combinations of $|L\rangle$ and $|R\rangle$. 5 pt

Leave your answer as 2 expressions in the simplest possible form.

Since every Jones vector can be written as a linear combination of $|H\rangle$ and $|V\rangle$ (see equation (3)), by doing the previous problem, you may realise that it is also possible to express every Jones vector as a linear combination of $|L\rangle$ and $|R\rangle$ instead. Similarly, it is also possible to show that every Jones vector is expressible as a linear combination of $|D\rangle$ and $|A\rangle$ as well. These are basically different coordinate systems for the space of Jones vectors.

Polarisers

With our new formalism for polarisation, we can start looking at a simple optical component. A **linear polariser** is an optical component that 'filters out' polarised light in a specific direction.



Only light aligned with the transmission axis is allowed to pass through the polariser. For example, for a vertically-aligned polariser, the x-component of the Jones vector will become 0, while the y-component remains the same. As such, we can write the action of the polariser as a matrix multiplication:

$$|J_{\text{aft}}\rangle = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} |J_{\text{bef}}\rangle$$

If $|J_{\text{bef}}\rangle$ is normalised, then the 'length' of $|J_{\text{aft}}\rangle$ will tell us the ratio of the amplitude of the final wave to the initial wave.

The matrix $M_V = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is known as the **Jones matrix** for the vertically-aligned polariser. Similarly, the Jones matrix for a horizontally-aligned polariser is:

$$M_H = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

A.4	Linear polarisers are not the only type of polarisers; **circular polarisers** also exist. Work out the Jones matrix for a right circular polariser, M_R. *Leave your answer in the simplest possible form.*	15 pt

Hint: For a right circular polariser, what will happen when you send in LCP and RCP light? Hence, what are the values of $M_R\,|R\rangle$ and $M_R\,|L\rangle$?

Section B. Spin (30 points)

Mathematical background

We have been writing column vectors using the notation $|X\rangle$. This object is known as a **ket**. Every ket column vector has a corresponding **bra** row vector that looks like $\langle X|$:

$$|X\rangle = \begin{pmatrix}\alpha \\ \beta\end{pmatrix} \leftrightarrow \langle X| = \begin{pmatrix}\overline{\alpha} & \overline{\beta}\end{pmatrix}$$

Remember that $\overline{z}$ is the complex conjugate of z.

With bras, we can now define the **inner product** of a bra and a ket[2]:

$$\langle A|B\rangle = (\alpha \;\; \beta)\begin{pmatrix}\gamma \\ \delta\end{pmatrix} = \alpha\gamma + \beta\delta$$

An inner product takes a bra and a ket, and produces a single complex number. Note that this works the same way as matrix multiplication. Furthermore, the inner product is linear; i.e. with μ and ν as scalars:

$$\langle A|\Big(\mu\,|B\rangle + \nu\,|C\rangle\Big) = \mu\langle A|B\rangle + \nu\langle A|C\rangle$$

$$\Big(\mu\,\langle A| + \nu\,\langle B|\Big)\,|C\rangle = \mu\langle A|C\rangle + \nu\langle B|C\rangle$$

Also note that:

$$\langle A|B\rangle = \overline{\langle B|A\rangle}$$

Thus, if you take any ket and compute the inner product with its corresponding bra, you get the square of the 'length' of the ket (see equation (4)):

$$\langle X|X\rangle = \begin{pmatrix}\overline{\alpha} & \overline{\beta}\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \overline{\alpha}\alpha + \overline{\beta}\beta = |\alpha|^2 + |\beta|^2$$

We will define $\sqrt{\langle X|X\rangle}$ as the **magnitude** of $|X\rangle$.

B.1 Recall the Jones vectors $|H\rangle$, $|V\rangle$ and $|R\rangle$ from earlier. Fill in the table with the values of each of the possible inner products. 5 pt

Leave your answers in the simplest possible form.

[2]This is why they are known as bras and kets; their inner product will form a 'braket'. If the bras and kets are composed of real-valued components, this inner product is also more commonly known as a **dot product**.

Spin is like (and unlike) polarisation

In quantum mechanics, particles such as electrons have an intrinsic quantity known as **spin**. Spin is just as fundamental to the electron as its mass or charge.

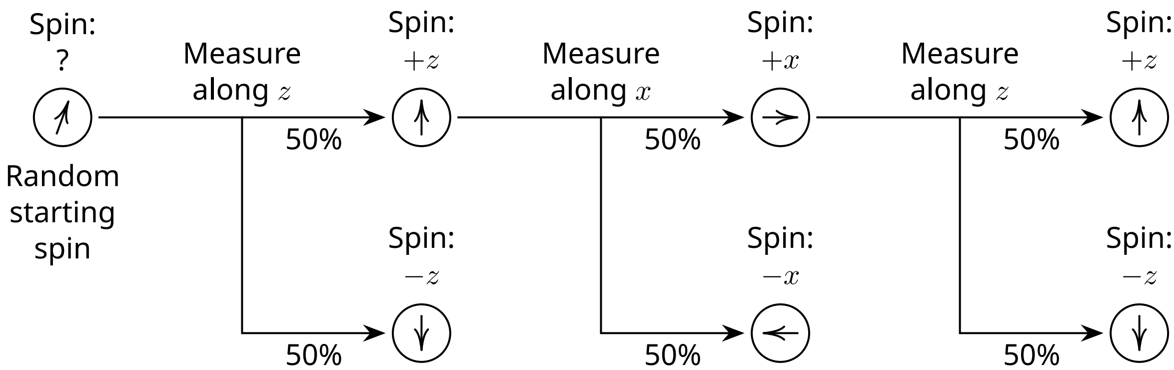
Due to the electron's quantum nature, if we measure an electron's spin along the z-axis, we will only ever get 2 possible states: up and down (or $+z$ and $-z$). We will represent these states as $|u\rangle$ and $|d\rangle$ respectively[3], and write:

$$|u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad |d\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In quantum mechanics, the very act of measuring the electron's spin changes its spin state. This is somewhat similar to how passing a light beam through a linear polariser to measure its polarisation, will itself change the polarisation of the beam. For example, if we had measured an unknown electron's spin along the z-axis as up, the electron's spin after the measurement would change to become $|u\rangle$.

However, this is where spin differs from polarisation: if an electron's state is currently $|u\rangle$ (i.e. the spin is along the $+z$ axis), and then we measure its spin along the x-axis, we do not get 0 spin! Since spins only have 2 possible states, the electron will instead have a 50% chance of being measured in the $+x$ state, and a 50% chance of being measured in the $-x$ state.

Finally, the electron does not 'remember' its previous spin. If we had measured the electron as $+x$ and then measured its spin along the z-axis, we still get a 50% chance of being measured either along $+z$ or $-z$.



[3]Similarly to Jones vectors, an electron's spin state is NOT an actual vector in 3D/2D space; it is a mathematical object containing 2 complex numbers as its components, that behaves like a vector in an abstract '**state space**'.

The behaviour of spins

If we define the $+x$ and $-x$ states as $|r\rangle$ (right) and $|l\rangle$ (left) respectively; and the $+y$ and $-y$ states as $|i\rangle$ (in) and $|o\rangle$ (out) respectively, then:

$$|r\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad |l\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$
$$|i\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} \qquad |o\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$

Notice the similarities between these and the Jones vectors! Such 2-component objects are known as **spinors**.

With the inner product, we have the tools we need to analyse the electron behaviour. If an electron is currently in state $|A\rangle$ and we measure its spin along some axis l that corresponds to states $|B_+\rangle$ and $|B_-\rangle$, the **probability** of measuring the electron state as $|B_+\rangle$ is[4]:

$$P(\text{state measured is } B_+) = |\langle A|B_+\rangle|^2$$

Note that if the electron is already in state $|B_+\rangle$ and we measure along l again, the probability of measuring $|B_+\rangle$ has to be 1, so $\langle B_+|B_+\rangle = 1$ and hence all spinors have to be normalised.

Now, with this new information, we can explain the electron's behaviour: when the electron is in the state $|u\rangle$ and we measure the electron's spin in the x-axis, we can calculate $|\langle u|r\rangle|^2$:

$$|\langle u|r\rangle|^2 = \left|(1\ \ 0)\,\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\right|^2 = \left|\frac{1}{\sqrt{2}}(1)\right|^2 = \frac{1}{2}$$

Since $|\langle u|r\rangle|^2 = 1/2$, the probability that we observe $+x$ is $1/2$, but this measurement also causes the electron's state to change into $|r\rangle$. This process repeats when we try measuring the state along the z-axis again.

B.2 Suppose we have a machine that produces many of electrons, all with the same spin state $|\psi\rangle$. After many of experiments on many different electrons that all start with the state $|\psi\rangle$, we have determined that when measuring spin about the **10 pt**

- x-axis, the probability of obtaining $|r\rangle$ (i.e. $+x$) is $P_{+x} = 0.50$.
- y-axis, the probability of obtaining $|i\rangle$ (i.e. $+y$) is $P_{+y} = 0.98$.
- z-axis, the probability of obtaining $|u\rangle$ (i.e. $+z$) is $P_{+z} = 0.36$.

Determine the state $|\psi\rangle$ as a normalised spinor. Ensure that the upper component of the spinor $|\psi\rangle$ is a nonnegative real number.

Leave your answer in the simplest possible form.

[4] Notice that if we multiply $|A\rangle$ by a factor of the form $e^{i\theta}$, $|(e^{-i\theta}\langle A|)\,|B_+\rangle|^2 = |\langle A|B_+\rangle|^2$, i.e. this has no effect on any of the actual *physics* of the problem and cannot be measured. Therefore, the usual convention is to use this phase factor to ensure that the coefficient of $|u\rangle$ is a nonnegative real number. In quantum field theory, it turns out that this 'symmetry' results in both classical electromagnetism and the conservation of electric charge.

Spin operators

It may be tempting to try to find a matrix that represents the change in the electron's spin state after a measurement; however, since the change in the state is probabilistic, this is not possible. We will work instead with matrices known as **operators**, which can help us represent physical observables (such as spin, momentum, or energy) in the system.

If an operator M represents an observable, and if λ is one possible outcome of measuring that observable with $|m\rangle$ being the corresponding state, M must satisfy:

$$M|m\rangle = \lambda|m\rangle \tag{5}$$

$|m\rangle$ is said to be an **eigenstate** of M, and λ is the corresponding **eigenvalue**. Note that if $|\psi\rangle$ is the state before the measurement, $M|\psi\rangle$ is **NOT** the state after the measurement; the state will become one of the eigenstates of M, such as $|m\rangle$.

For example, consider the spin of the electron along the z-axis, which is an observable that can be measured. The actual numerical value of an electron's spin is always $\pm\frac{\hbar}{2}$, where $\hbar = \frac{h}{2\pi}$ is the reduced Planck's constant. As such, the measurement will yield $+\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$, corresponding to states $|u\rangle$ and $|d\rangle$ respectively.

Therefore, the spin operator in the z-direction S_z must have $|u\rangle$ and $|d\rangle$ as its eigenstates, and $+\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$ as the corresponding eigenvalues. S_z thus satisfies equation (5):

$$S_z|u\rangle = \frac{\hbar}{2}|u\rangle \qquad \text{and} \qquad S_z|d\rangle = -\frac{\hbar}{2}|d\rangle$$

From this, S_z can be calculated to be:

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar}{2}\sigma_z \tag{6}$$

Similarly, we can also calculate S_x and S_y, the operators representing the spin of the electron along the x- and y-axes respectively:

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_x \qquad \text{and} \qquad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_y \tag{7}$$

The matrices σ_x, σ_y and σ_z are known as the **Pauli matrices**.

At this point, we can now construct the spin operator for *any* direction $\hat{\mathbf{n}}$ (where $\hat{\mathbf{n}}$ is a unit vector pointing in some direction[5]):

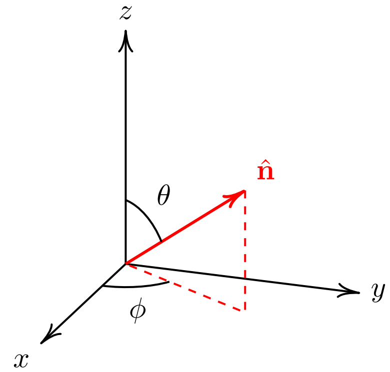
$$S_n = \hat{\mathbf{n}} \cdot \mathbf{S} = n_x S_x + n_y S_y + n_z S_z \tag{8}$$

The notation $\hat{\mathbf{n}} \cdot \mathbf{S}$ is just a notational shorthand, and is not an actual dot product[6].

[5] $\hat{\mathbf{n}}$ is an actual physical vector that points in some direction in 3D space, unlike objects such as $|u\rangle$. For the purposes of this problem, all the physical vectors will be in **lowercase boldface** (e.g. $\mathbf{a}$), while all the objects such as $|u\rangle$ that only exist in an abstract state space will be in bra/ket notation.

[6] $\mathbf{S} = S_x\hat{\mathbf{x}} + S_y\hat{\mathbf{y}} + S_z\hat{\mathbf{z}}$ is an object that looks like a 3-dimensional vector, but its components are actually matrices! In fact, a physicist would say that $\mathbf{S}$ has 1 tensor index and 2 spinor indices.

B.3 Consider a detector that is oriented in an arbitrary direction. In spherical coordinates, $\hat{\mathbf{n}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$, where $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$. 15 pt



Let the spin operator for this direction be S_n. Calculate the eigenstate $|\chi\rangle$ of S_n that corresponds to the $+\frac{\hbar}{2}$ eigenvalue (i.e. spin up along $\hat{\mathbf{n}}$). Leave your answer as a normalised spinor, with the upper component being a nonnegative real number.

Leave your answer in the simplest possible form in terms of θ and ϕ.

Hint:

$$\tan\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}$$

Section C. Transformations (40 points)

Mathematical background

Matrix multiplication:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

Matrix exponentiation: for any square matrix M, the exponent is defined as:

$$e^M = I + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \cdots = I + \sum_{k=1}^{\infty} \frac{1}{k!}M^k$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix, where multiplying it by any matrix or vector leaves the matrix or vector unchanged.

Useful infinite series:

$$\begin{aligned} e^x &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots \\ \sin x &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \\ \cos x &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots \\ \sinh x &= x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots \\ \cosh x &= 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \cdots \end{aligned}$$

where

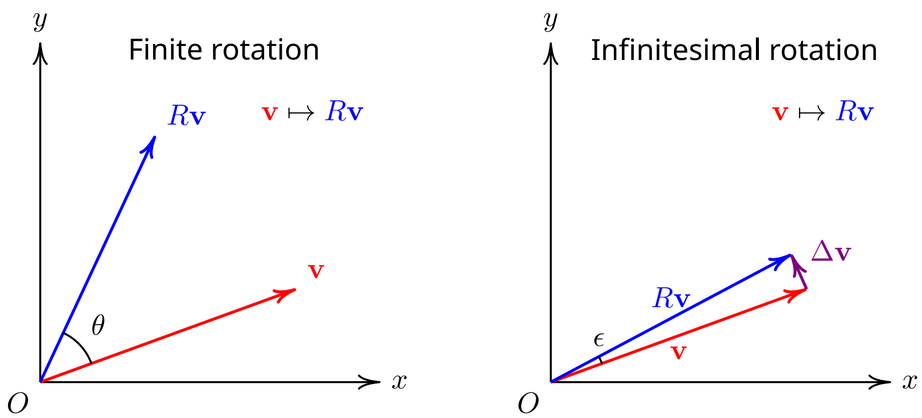
$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Infinitesimal transformations

Given any object $\mathbf{v}$, to transform it, we can apply a transformation M:

$$\mathbf{v} \mapsto M\mathbf{v}$$

For example, consider the act of rotating a 2-dimensional vector $\mathbf{v}$:



The transformation for the rotation, $R(\theta)$, is clearly a matrix of some kind that is a function of θ. We will use a powerful method, provided by **Lie theory**, to determine R.

Instead of considering a finite rotation by an angle of θ, we consider an infinitesimal rotation by an angle of ϵ.

For very small ϵ, the vector $\Delta\mathbf{v}$ is approximately perpendicular to $\mathbf{v}$. To make a perpendicular vector to $\mathbf{v}$, the transformation is[7]:

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Note that by geometry, $\Delta\mathbf{v}$ needs to have a length of $\epsilon|\mathbf{v}|$, so:

$$\Delta\mathbf{v} = \epsilon J\mathbf{v}$$

The total transformation matrix is thus:

$$R(\epsilon)\mathbf{v} = \mathbf{v} + \Delta\mathbf{v} = (I + \epsilon J)\mathbf{v}$$

$$\implies R(\epsilon) = I + \epsilon J$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix.

This result makes sense: for small ϵ, we expect $R(\epsilon)$ to be almost identical to I, except for a small term that is first order in ϵ. We call J the **generator** of the transformation. With the generator, we can now calculate R for finite rotations (not just infinitesimal rotations) using a result from Lie theory known as the **exponential map**[8]:

$$R(\theta) = e^{\theta J}$$

To calculate $e^{\theta J}$, observe that $J^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$. Hence,

$$\begin{aligned} R(\theta) &= I + \theta J + \frac{1}{2!}(\theta J)^2 + \frac{1}{3!}(\theta J)^3 + \cdots \\ &= \left(I + \frac{\theta^2}{2!}J^2 + \frac{\theta^4}{4!}J^4 + \cdots\right) + \left(\theta J + \frac{\theta^3}{3!}J^3 + \frac{\theta^5}{5!}J^5 + \cdots\right) \\ &= I\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + J\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right) \\ &= I\cos\theta + J\sin\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \end{aligned}$$

We have thus managed to calculate the 2×2 rotation matrix by considering infinitesimal transformations. The table below shows the relationship between generators and transformation matrices:

[7] One way to see this is to think about the components of $\mathbf{v}$: the x-component needs to become the new y-component, while the y-component needs to become the negative of the new x-component.

[8] Here, we are using the mathematics convention rather than the physics one since it is less confusing. As such, many of the equations in the rest of the problem may look different from the literature.

<table>
<tr><th>Infinitesimal transformations</th><th>Finite transformations</th></tr>
<tr><td colspan="2">Equation of transformation: $$\mathbf{v} \mapsto R\mathbf{v}$$</td></tr>
<tr><td>$R = I + \epsilon J$</td><td>$R = e^{\theta J}$</td></tr>
<tr><td colspan="2">$$\begin{tikzcd} \underset{\text{generators}}{J} \arrow[r, bend left, "\text{Generates via exponential map}"] & \underset{\text{transformations}}{R} \arrow[l, bend left, "\text{Calculate in limit of small } \theta"] \end{tikzcd}$$</td></tr>
</table>

C.1 Recall the spinor from problem **B.3**, with $|\chi\rangle = \binom{\cos(\theta/2)}{e^{i\phi}\sin(\theta/2)}$. Let J_x, J_y and J_z be the generators for the rotation[a] of the spinor about their respective axes (following the right-hand rule). Let $\mathbf{J} = J_x\hat{\mathbf{x}} + J_y\hat{\mathbf{y}} + J_z\hat{\mathbf{z}}$ and $\boldsymbol{\sigma} = \sigma_x\hat{\mathbf{x}} + \sigma_y\hat{\mathbf{y}} + \sigma_z\hat{\mathbf{z}}$ (refer to equations (6) and (7) in the previous section for the definition of $\boldsymbol{\sigma}$, the Pauli matrices), then $\mathbf{J} = k\boldsymbol{\sigma}$ for a constant k. Determine k. **10 pt**

Leave your answer in the simplest possible form.

[a] Note that a rotation of the spinor corresponds to rotating its corresponding spin operator, then calculating the new eigenstate. However, computing this entire process is not required for this problem.

Hint: Spinors do **not** transform the same way as 2-dimensional vectors! Consider an infinitesimal rotation and its effect on $|\chi\rangle$ and use special cases for the value of ϕ if necessary. Recall that spinors can be multiplied by a phase factor of the form $e^{i\alpha}$ while still representing the same spin.

With our rotation generators $\mathbf{J}$ for spinors, the most general rotation matrix[9] for any spinor is $R(\boldsymbol{\theta}) = e^{\boldsymbol{\theta}\cdot\mathbf{J}}$ where $\boldsymbol{\theta}$ is a vector with the rotation angle as its magnitude and the rotation axis as its direction. The exponent $\boldsymbol{\theta}\cdot\mathbf{J}$ is defined similarly to equation (8).

The Lorentz boost

In special relativity, the Lorentz boosts[10] provide a way to transform a physical quantity from

[9] The set of such matrices have a special name: SU(2).

[10] You may have heard of a Lorentz transformation before, which is technically a combination of a rotation and a Lorentz boost. Hence, Lorentz boosts are a special case of the Lorentz transformation, although some sources use the terms interchangeably.

one velocity to another. For example, for a Lorentz boost along the x-direction,

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Here, ϕ is the **rapidity**, and is related to the velocity v by $\tanh\phi = v/c = \beta$. Note that the Lorentz boost will transform objects known as **4-vectors**. Also note that $\cosh\phi = \gamma$ and $\sinh\phi = \gamma\beta$, where $\gamma = 1/\sqrt{1-\beta^2}$ is the well-known **Lorentz factor**.

Note that for this problem, we are taking the **active view** of transformations: just like how our rotation transformations previously were physically rotating a vector (rather than rotating a coordinate system), our Lorentz boosts here are **physically boosting a 4-vector rather than changing reference frames** (e.g. applying the Lorentz boost to a momentum 4-vector would give you the new momentum 4-vector if the particle gained an extra velocity v in its current frame).

Just like rotations, Lorentz boosts also have generators. Let the generators for the Lorentz boosts in the x, y and z-directions be K_x, K_y and K_z respectively. For an arbitrary Lorentz boost, the transformation matrix is:

$$\Lambda = e^{\boldsymbol{\phi}\cdot\mathbf{K}}$$

where $\mathbf{K} = K_x\hat{\mathbf{x}} + K_y\hat{\mathbf{y}} + K_z\hat{\mathbf{z}}$. The magnitude of $\boldsymbol{\phi}$ is the rapidity of the transformation, while the direction represents the direction of the velocity of the new frame.

Bispinors

Spinors, of course, have different behaviour under Lorentz boosts than 4-vectors. For spinors, the generators for the Lorentz boost turn out to be[11]:

$$\mathbf{K} = \pm\frac{\boldsymbol{\sigma}}{2}$$

There are 2 possible ways that a spinor can transform under a Lorentz boost! As it turns out, there are in fact two types of ways that spin can transform in physics: some transform via the $+$ sign, while others transform via the $-$ sign.

We can define a **bispinor**[12] by stacking together 2 spinors:

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \psi_L^{\uparrow} \\ \psi_L^{\downarrow} \\ \psi_R^{\uparrow} \\ \psi_R^{\downarrow} \end{pmatrix}$$

[11] You may be wondering where these come from. If we define the **commutator** of 2 matrices as $[A,B] = AB - BA$, it is possible to calculate all the possible commutators among the generators for 4-vectors (e.g. $[K_x, K_y] = -J_z$, $[J_x, K_y] = K_z$, etc.). These are known as **commutation relations**. According to Lie theory, these relationships must also hold for the generators for *spinors*, since they represent the same underlying physical transformations. Since we already know J_x, J_y and J_z for spinors, it is a matter of solving those commutation relations for K_x, K_y and K_z.

[12] Also known as a **Dirac spinor** (while the 2-component spinors we have been considering so far are **Weyl spinors**).

where $\psi_L = \binom{\psi_L^\uparrow}{\psi_L^\downarrow}$ and $\psi_R = \binom{\psi_R^\uparrow}{\psi_R^\downarrow}$ are the two types of spinors (ψ_L transforms with $-\frac{1}{2}\boldsymbol{\sigma}$, while ψ_R transforms with $+\frac{1}{2}\boldsymbol{\sigma}$). The transformation of a bispinor under a Lorentz boost is therefore:

$$\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \mapsto \begin{pmatrix} e^{-\boldsymbol{\sigma}\cdot\boldsymbol{\phi}/2} & 0 \\ 0 & e^{\boldsymbol{\sigma}\cdot\boldsymbol{\phi}/2} \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

How do we physically interpret a bispinor? They represent particles known as **Dirac particles**, and electrons are one such example. It turns out that electrons are actually a mixture of the ψ_L and ψ_R spinors! In our old notation, if the spin state of the electron is $|\chi\rangle = \xi$, then the state of the electron **at rest** is described by:

$$\psi = \sqrt{mc^2} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

where m is the mass of the electron and c is the speed of light (we now have an extra factor of $\sqrt{mc^2}$ by convention). Note that since each spinor no longer fully describes the state of the electron, we have shifted away from using bra-ket notation, in favour of greek letters such as ξ. The corresponding bra of a spinor ξ will now be known as the **adjoint**, denoted $\xi^\dagger$.

C.2 An electron at rest in the spin up state is described by the bispinor: 15 pt

$$\psi = \sqrt{mc^2} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \sqrt{mc^2} \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix}$$

where $\xi_1 = |u\rangle = \binom{1}{0}$. What would be the new bispinor describing the electron if it was moving with some velocity v in the $+z$ direction, in terms of the rapidity $\phi = \tanh^{-1}(v/c)$?

Leave your answer in the simplest possible form in terms of ϕ.

When the electron is at rest, our old normalisation conditions still hold, with $\psi_L = \psi_R = \sqrt{mc^2}\,\xi$ and $\psi_L^\dagger \psi_L = \psi_R^\dagger \psi_R = mc^2 \xi^\dagger \xi = mc^2$. However, for moving electrons, the normalisation conditions on the individual spinors no longer hold. Instead, there is now a new normalisation condition involving both components of the bispinor[13].

C.3 Let the bispinor of an electron with an arbitrary velocity be $\psi = \binom{\psi_L}{\psi_R}$. Write down the normalisation condition(s) on the bispinor ψ in terms of ψ_L, ψ_R and their corresponding adjoints $\psi_L^\dagger$ and $\psi_R^\dagger$. 10 pt

Leave your answer in the simplest possible form in terms of ψ_L, ψ_R, $\psi_L^\dagger$ and/or $\psi_R^\dagger$.

[13] For those familiar with 4-vectors, this is similar to the idea of a Lorentz invariant; however bispinors are not 4-vectors and therefore the normalisation condition is different.

Hint: Your answer in **C.2** should be normalised. Also remember that you can only form a scalar by combinations of the form $\chi^\dagger \zeta$ (this is $\langle \chi | \zeta \rangle$ in our old notation). The order matters!

Note that any bispinor state can be written as a combination of:

$$\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \sqrt{mc^2} \begin{pmatrix} \xi \\ \xi \end{pmatrix} + \sqrt{mc^2} \begin{pmatrix} \eta \\ -\eta \end{pmatrix}$$

We know that the left term represents electrons, but what does the right term represent? As it turns out, it represents antielectrons (or **positrons**) at rest, which are particles identical to electrons, except with the opposite charge. Hence, we have "predicted" the existence of antimatter! A similar argument was used by Paul Dirac in 1928 to predict their existence, which was experimentally verified in 1932[14].

The Universe is left-handed

For a particle travelling with momentum in the direction of a unit vector $\hat{\mathbf{p}}$, define the **helicity** operator[15] as:

$$\mathcal{H} = \begin{pmatrix} -\hat{\mathbf{p}} \cdot \boldsymbol{\sigma} & 0 \\ 0 & \hat{\mathbf{p}} \cdot \boldsymbol{\sigma} \end{pmatrix}$$

C.4	Consider the electron from part **C.2**. In the ultrarelativistic limit (i.e. v approaching c), the state of the electron becomes an eigenstate of the helicity operator. Find the corresponding eigenvalue λ. *Leave your answer in the simplest possible form.*	5 pt

The helicity operator tells us whether the spin of the electron is aligned with its momentum. In particular, this is the reason for the choice of the subscripts for ψ_L and ψ_R: they represent left- and right-handed particles.

Note that for massive particles, helicity is not an inherent property (since you can always change to a reference frame where the particle's direction of motion is reversed, but spin remains the same). However, for massless particles, it becomes an inherent property since there is no reference frame where the particle is not moving at the speed of light. In particular, for photons[16], left- and right-handed photons correspond to LCP and RCP light (the Jones vectors $|L\rangle$ and $|R\rangle$ from earlier)!

Does helicity have any physical significance for other particles? In the theory of the weak interaction, interestingly, there is a field $W_\mu(x)$ that *only* affects left-handed Dirac particles! As such, Nature is inherently "left-handed" at the fundamental level.

[14] Why is our bispinor now representing 2 different particles? This is because we have been treating electrons (and positrons) as particles, when they really should have been treated as waves. In fact, electrons and positrons are just oscillations in the spinor field ψ, much like how photons are oscillations in the electromagnetic field. Interestingly, when the particles are moving, the matter and antimatter terms start to "bleed into" each other, demonstrating how the existence of antimatter is related to special relativity.

[15] This is defined slightly differently in the literature as well.

[16] Note that since photons have a different spin representation compared to electrons (photons are spin-1, while electrons are spin-$\frac{1}{2}$), they are not Dirac particles. However, they do still have spin and thus still have helicities.